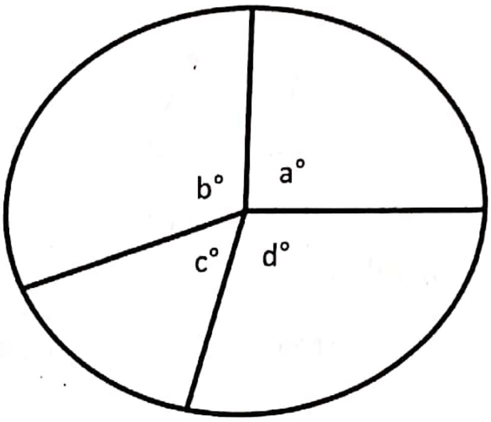
Angles c and f are opposite angles. Therefore, m∠c equals m∠f. The measure of ∠f must be 35°.

Rule 3: The sum of all the measures of all the angles around a point is 360°.



a + b + c + d = 360

POLYGON

A polygon is a two-dimensional figure with three or more straight sides. Polygons are named according to the number of sides they have.

Number of Sides	Name
3	Triangle
4	Quadrilateral
5	Pentagon
n	n-gon

Characteristics of a Polygon

All polygons, no matter how many sides they possess, share certain characteristics.

- The sum of the interior angles of a polygon with n sides is (n-2)180°. For instance, the sum of the interior angles of a triangle is (3-2)180°= 1(180°) = 180°.
- A polygon with equal sides and equal interior angles is a regular polygon.
- The sum of the exterior angles of any polygon is 360°.
- The perimeter of a polygon is the sum of the lengths of its sides.
- The area of a polygon is the measure of the area of the region enclosed by the polygon. Each polygon tested on the General Ability Tests has its own unique area formula.

TRIANGLE

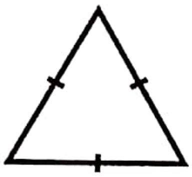
A **triangle** is a closed figure with three sides.

Classification of Triangles

Triangle can be classified according to their sides or interior angles as shown below.

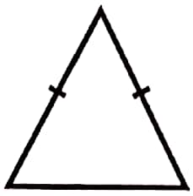
Classification of triangles by sides

Equilateral



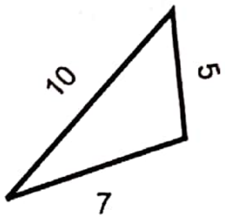
3 sides equal

Isosceles



2 sides equal

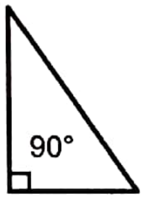
Scalene



No sides equal

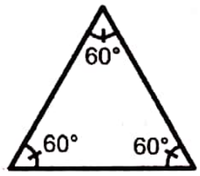
Classification of triangles by angles

Right



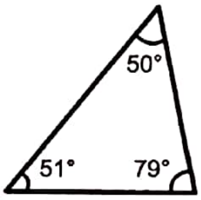
One angle exactly 90°

Equiangular



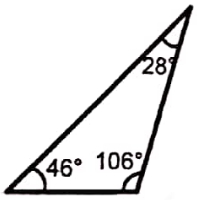
3 angles equal

Acute



All 3 angles
less than 90°

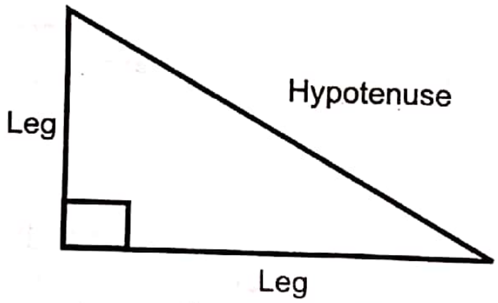
Obtuse



One angle
more than 90°

Right Triangle

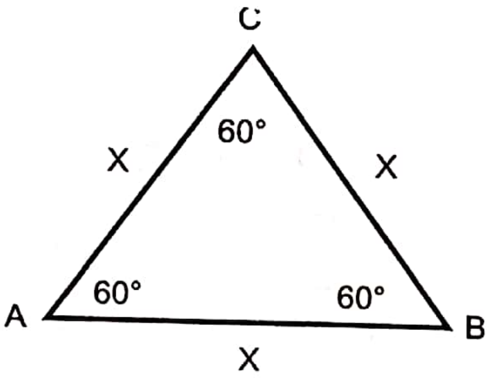
A **right triangle** has two sides that form a right 90° angle. These two sides are called the **legs** of the triangle. The side opposite the right angle is called the **hypotenuse.**



Please note that the right angle is usually indicated by a rectangle drawn between the legs.

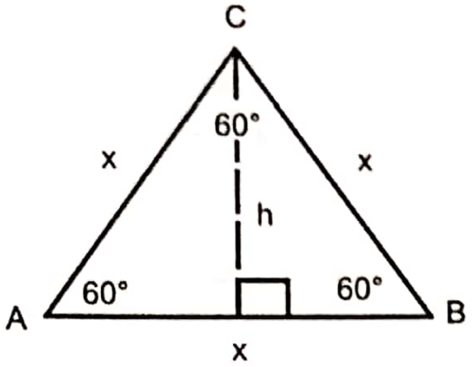
Equilateral Triangle

An equilateral triangle is a triangle with three equal sides. The three angles of an equilateral triangle are also equal. Each angle measures 60° .

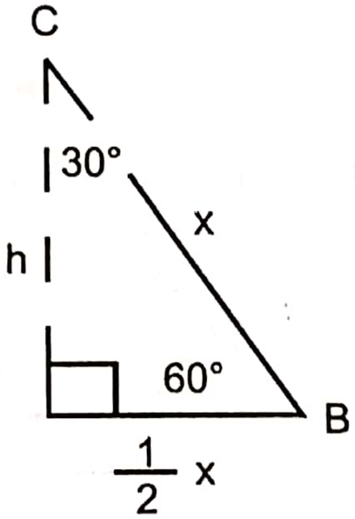


In the case of an equilateral triangle, the altitude bisects both the opposite side and the angle from whose vertex it originated.

In the triangle below, the altitude h divides side **x** and ∠ACB into two identical halves. Incidentally, the line drawn from any vertex to the opposite side is called altitude or height of the triangle, if and only if, the two lines form a right angle.

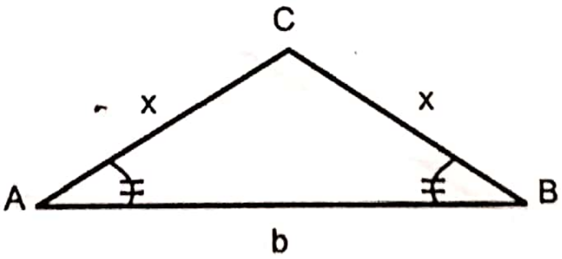


From the equilateral triangle we can derive a triangle, which is sometimes referred to as the **30- 60- 90 Triangle**, by drawing the altitude from any vertex.

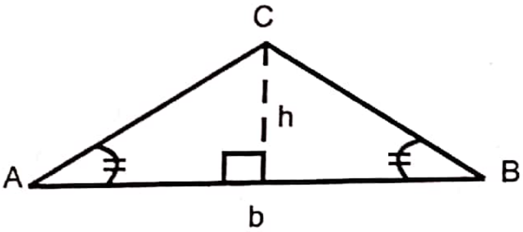


Isosceles Triangle

An isosceles triangle is a triangle with exactly two equal sides. The angles opposite these sides are also equal. In the triangle below, ∠A equals ∠B .

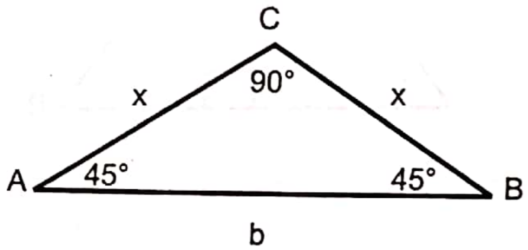


In the case of an isosceles triangle, if and only if the altitude is drawn from the angle that is not equal to the other two, it bisects both the opposite side and the angle from whose vertex it originated. In the triangle below, the altitude **h** divides side **b** and angle **c** into two identical halves.



Right Isosceles Triangle

A right isosceles triangle is a triangle with exactly two equal sides. The angles opposite these sides each measure exactly 45°. This means that the third angle measure must be 90°. This triangle is sometimes referred to as the **45- 45- 90 Triangle.**

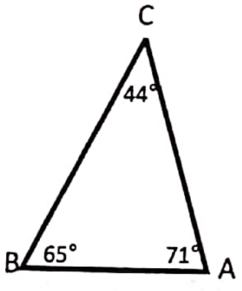


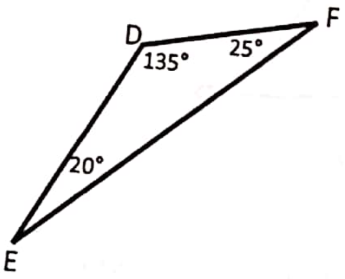
Scalene Triangle

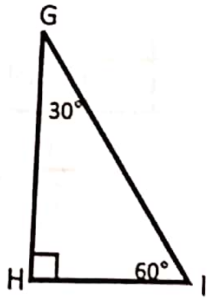
A triangle having sides of different lengths and angles of different measure. According to this definition, a right triangle can also be called a scalene triangle.

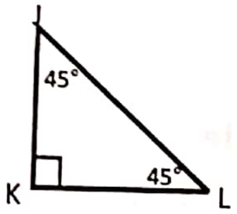
RULES FOR TRIANGLES

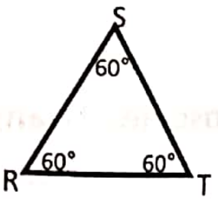
1. The Rule of Interior Angles. *The sum of the interior angles of a triangle always equals 180°.* Whenever you're given two angles of a triangle, you can calculate the third angle. For example:



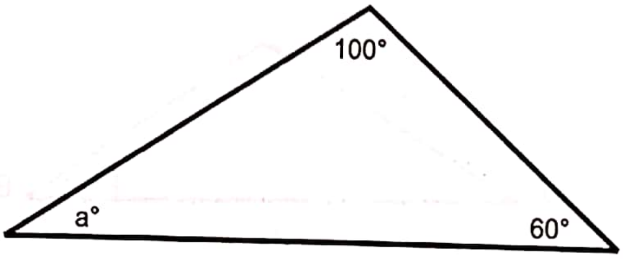






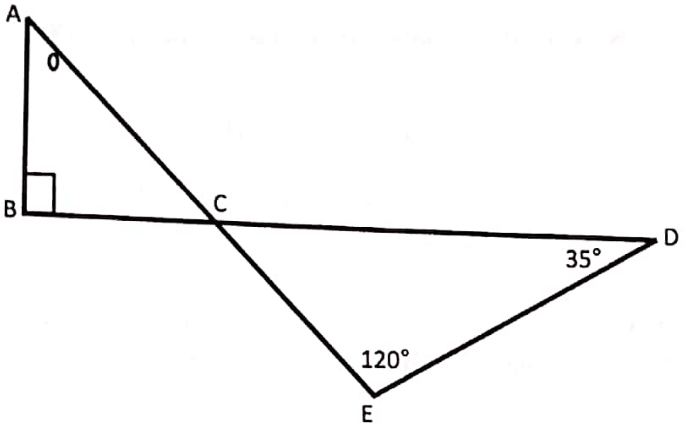


Example1: What is the value of a in the figure below?



Since the sum of the angles of a triangle equals 180°, you can set up an equation: a° + 100° + 60° = 180°. Isolating the variable and solving for a gives a = 20.

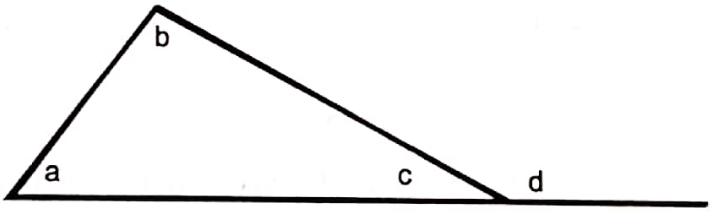
Example2: In the figure below, what is the value of $\angle BAC$?



Use the above rule twice:

- m ∠DCE + 120 + 35 = 180 =>
- *m ∠DCE*+ 155 = 180 => *m ∠DCE* = 25.
- Since vertical angles are equal, *m ∠ACB*= 25
- ∠ABC is right angle that is equal to 90
- ∠BAC + 90 + 25 = 180 => ∠BAC + 115 = 180 => ∠BAC = 65.

2. The Rule of Exterior Angles. An exterior angle of a triangle is the angle formed by extending one of the sides of the triangle past a vertex (or the intersection of two sides of a figure). In the figure below, d is the exterior angle.

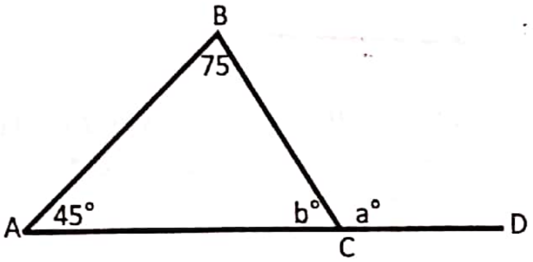


d + c = 180
d = a + b

Since, together, d and c form a straight angle, they add up to 180° i.e. d + c = 180°. According to the first rule of triangles, the three angles of a triangle always add up to 180°, so a + b + c = 180°. Since d + c = 180° and a + b + c = 180°, d must be equal to a + b (the remote interior angles).

This generalizes to all triangles as the following rules: The exterior angle of a triangle plus the interior angle with which it shares a vertex is always 180°. *The exterior angle is also equal to the sum of the measures of the remote interior angles.*

Example 3: In the figure below, what is the value of a?

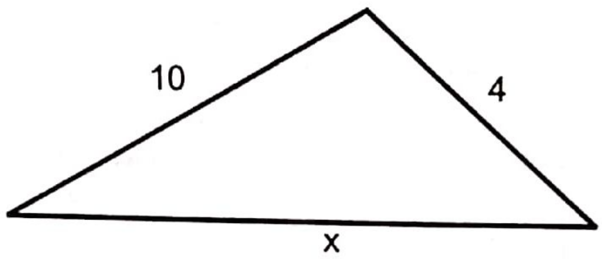


The exterior angle is also equal to the sum of the measures of the remote interior angles therefore

$$\begin{aligned} a^\circ &= \angle A + \angle B \\ &= 45 + 75 \\ &= 120 \end{aligned}$$

3. The Rule of the Sides. *The length of any side of a triangle must be greater than the difference and less than the sum of the other two sides.* In other words, difference of other two sides < one side < sum of other two sides.

Although this rule won't allow you to determine a precise length of the missing side, it will allow you to determine a range of values for the missing side, which is exactly what the test makers would ask for in such a problem. Here's an example:



What is the range of values for x in the triangle above?

Since the difference of 10 and 4 is 6, and the sum of 10 and 4 is 14, we can determine the range of values of x:

$$6 < x < 14$$

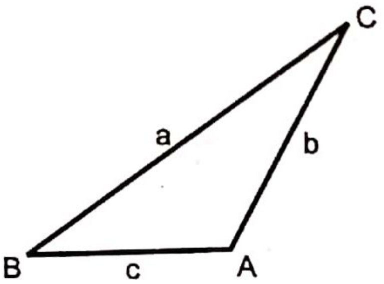
Keep in mind that x must be inside this range. That is, x could not be 6 or 14.

Example 4: Which of the following triangle is not possible?

(a) 3, 5, 7 **(b)** 4, 7, 12 **(c)** 6, 8, 10 **(d)** 18, 22, 27 **(E)** 9, 14, 17

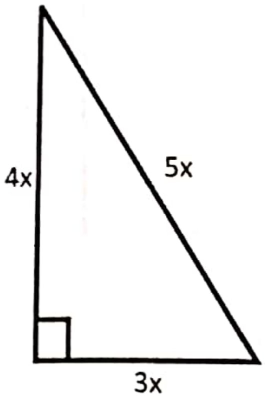
Explanation: Sum of two sides 4 and 7 is less than 12, therefore it is not a possible triangle.

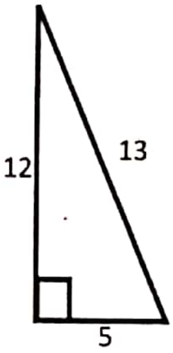
4. The Rule of Proportion. *In every triangle, the longest side is opposite the largest angle and the shortest side is opposite the smallest angle.* Take a look at the following figure and try to guess which angle is largest.

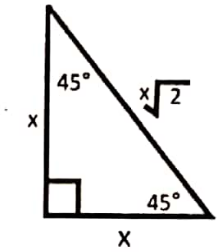


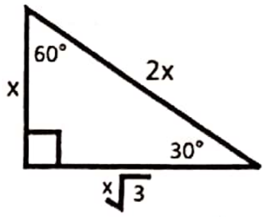
In this figure, side a is clearly the longest side and $\angle A$ is the largest angle. Meanwhile, side c is the shortest side and $\angle C$ is the smallest angle. So $c < b < a$ and $\angle C \angle B \angle A$ .This proportionality of side lengths and angles holds true for all triangles.

Below are the **right triangles that appear most often on the examination.** You should recognize them immediately whenever they come up in questions. Carefully study each one, and memorize.









On the examination, the most common right triangles whose sides are integers are the 3 – 4 - 5 triangle.

Example 5: Find the length of the hypotenuse of a right triangle if the lengths of the other two sides are 6 inches and 8 inches.

Solution: Step 1: Test the ratio of the lengths to see if it fits the 3n : 4n : 5n ratio.

$6 : 8 : ? = 3(2) : 4(2) : ?$

Step 2: Yes, it is a 3-4-5 triangle for n = 2.

Step 3: Calculate the third side

$5n = 5 \times 2 = 10$

The length of the hypotenuse is 10 inches.

Example 6: Find the length of one side of a right triangle if the length of the hypotenuse is 15 inches and the length of the other side is 12 inches.

Solution: Step 1: Test the ratio of the lengths to see if it fits the 3n : 4n : 5n ratio.

$? : 12 : 15 = ? : 4(3) : 5(3)$

Step 2: Yes, it is a 3-4-5 triangle for n = 3

Step 3: Calculate the third side

$3n = 3 \times 3 = 9$

The length of the side is 9 inches.

Rule 5. For any positive number *x*, there is a right triangle whose sides are 3x, 4x, 5x.

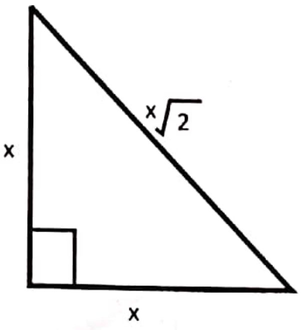
For example:

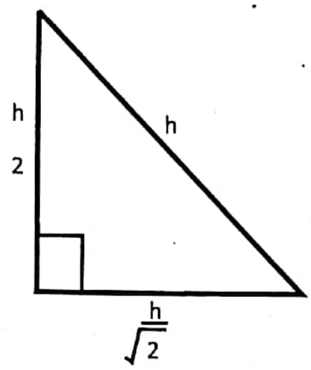
x= 1	3, 4, 5	x = 5	15, 20, 25
x=2	6, 8, 10	x= 10	30, 40, 50
x=3	9, 12, 15	x= 50	150, 200, 250
x=4	12, 16, 20	x= 100	300, 400, 500

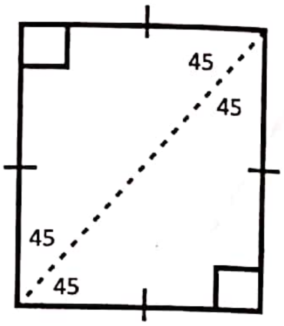
Rule 6. In a 45 – 45 - 90 right triangle, the sides are *x*, *x*, and $x\sqrt{2}$ So,

- by multiplying the length of a leg by $\sqrt{2}$, you get the hypotenuse.

- by dividing the hypotenuse by$\sqrt{2}$, you get the length of each leg.







Example 7: Find the lengths of the other two sides of a right triangle if the length of the hypotenuse is $4\sqrt{2}$ inches and one of the angles is 45°.

Solution: Step 1: This is a right triangle with a 45° so it must be a 45°-45°-90° triangle.

Step 2: You are given that the hypotenuse is $4\sqrt{2}$. If the third value of the ratio $n : n : n\sqrt{2}$ is $4\sqrt{2}$ then the lengths of the other two sides must 4.

The lengths of the two sides are both 4 inches.

Example 8: Find the length of the hypotenuse of a right triangle if the lengths of the other two sides are both 3 inches.

Solution: Step 1: This is a right triangle with two equal sides so it must be a 45°-45°-90° triangle.

Step 2: You are given that the both the sides are 3. If the first and second value of the ratio $n : n : n\sqrt{2}$ is 3 then the length of the third side is $3\sqrt{2}$

The length of the hypotenuse is $3\sqrt{2}$ inches.

Rule 7. In a 30 – 60 - 90 right triangle the sides are x, $x\sqrt{3}$, and 2x

If you know the length of the shorter leg (x),

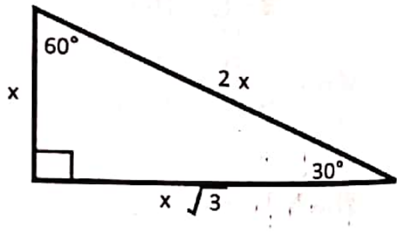
- multiply it by $\sqrt{3}$ to get the longer leg, and
- multiply it by 2 to get the hypotenuse.

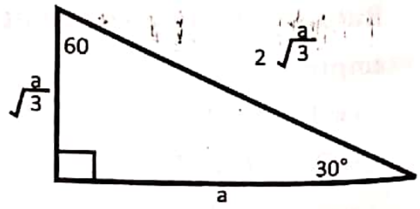
If you know the length of the longer leg *(a)*,

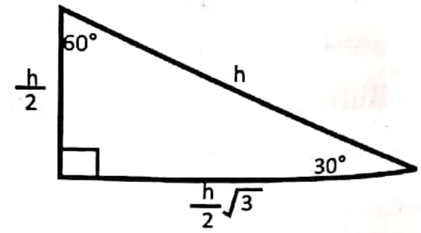
- divide it by $\sqrt{3}$ to get the shorter leg, and
- multiply the shorter leg by 2 to get the hypotenuse.

If you know the length of the hypotenuse *(h)*

- divide it by 2 to get the shorter leg, and
- multiply the shorter leg by $\sqrt{3}$ to get the longer leg.







Example 9: Find the length of the hypotenuse of a right triangle if the lengths of the other two sides are 4 inches and $4\sqrt{3}$ inches.

Solution: Step 1: Test the ratio of the lengths to see if it fits the $n{:}n\sqrt{3}{:}2n$ ratio

$$4 : 4\sqrt{3} : ? = n : n\sqrt{3} : 2n$$

Step 2: Yes, it is a 30-60-90 triangle with $n = 4$

Step 3: Calculate the third side.

$2n = 2 \times 4 = 8$

The length of the hypotenuse is 8 inches.

You can also recognize a 30°-60°-90° triangle by the angles. As long as you know that one of the angles in the right-angle triangle is either 30° or 60° then it must be a 30°-60°-90° special right triangle. A right triangle with a 30°-angle or 60°-angle must be a 30°-60°-90° special right triangle.

Example 10: Find the lengths of the other two sides of a right triangle if the length of the hypotenuse is 8 inches and one of the angles is 30°.

Solution: This is a right triangle with a 30°-60°-90° triangle.

You are given that the hypotenuse is 8. Substituting 8 into the third value of the ratio $n{:}n\sqrt{3}:2n$, we get that $2n = 8$ and $n=4$. Substituting $n = 4$ into the first and second value of the ratio we get that the other two sides are 4 and $4\sqrt{3}$. The lengths of the two sides are 4 inches and $4\sqrt{3}$ inches.

PYTHAGOREAN THEOREM

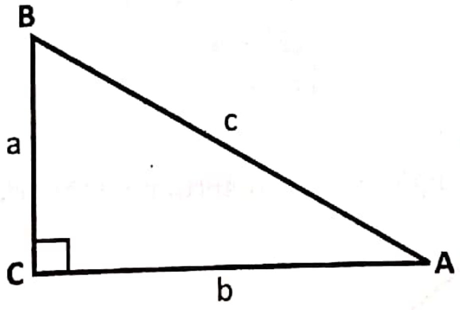
The Pythagorean Theorem is a fundamental relation in geometry among the three sides of a **right triangle.** It states that the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides.

The theorem can be written as an equation relating the lengths of the sides a, b and c, often called the "Pythagorean equation":

$$a^2 + b^2 = c^2$$

Where c represents the length of the hypotenuse and a and b the lengths of the triangle's other two sides.

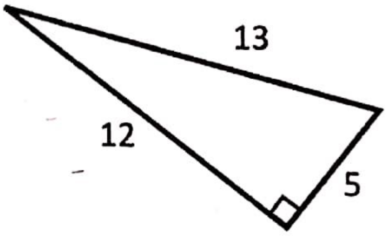
Let a, b, and c be the sides of ΔABC, with $a \le b \le c$.



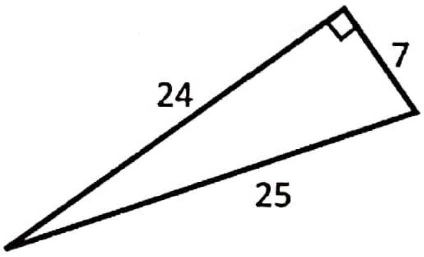
- $a^2 + b^2 = c^2$ if and only if angle C is a right angle. (*ΔABC is* a right triangle.)
- $a^2 + b^2 < c^2$ if and only if angle C is obtuse. (*ΔABC is* an obtuse triangle.)
- $a^2 + b^2 > c^2$ if and only if angle C is acute.(*ΔABC*is an acute triangle.)

When you use the Pythagorean theorem, just remember that the hypotenuse is always 'C' in the formula above . Look at the following examples:

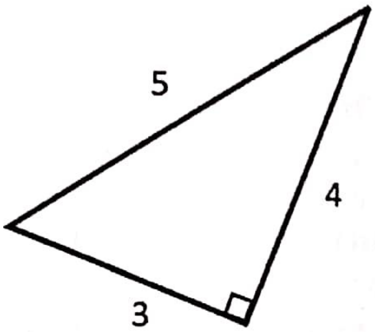
$5^2 \div 12^2 = 13^2$



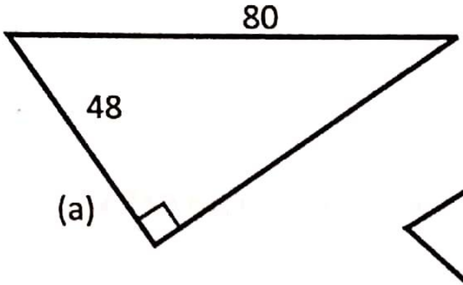
$24^2 \div 7^2 = 25^2$

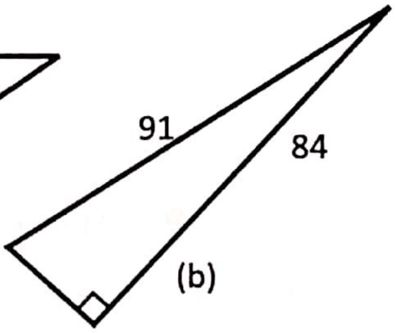


$4^2 \div 3^2 = 5^2$



Example 1: Given the right triangles displayed below, find the lengths of the remaining sides.





(a) The side opposite the right angle is c; how I name the other two sides really doesn't matter. I'll plug the values into the Theorem, and solve:

$$80^2 = 48^2 + b^2$$

$$6400 = 2304 + b^2$$

$$4096 = b^2$$

$$64 = b$$

(b) This triangle one works the same way as did **(a)**:

$$91^2 = 84^2 + b^2$$

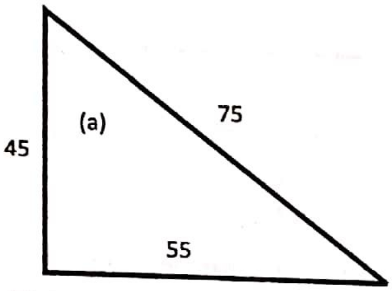
$$8281 = 7056 + b^2$$

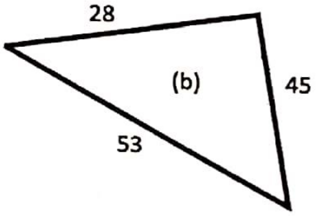
$$1225 = b^2$$

$$35 = b$$

$$b = 35$$

Example 2: Given the triangles below, determine if the triangles are right.





(a) I need to see if the squares of the legs equal the square of the hypotenuse:

$$45^2 + 55^2 = 2025 + 3025 = 5050$$

$$75^2 = 5625$$

The triangle for **(a)** is NOT a right triangle.

(b) This one works the same as for **(a)**:

$$28^2 + 45^2 = 784 + 2025 = 2809$$

$$53^2 = 2809$$

The triangle for **(b)** is a right triangle.

AREA OF A TRIANGLE

Area is a measure of the space covered by a two dimensional region.

Methods for finding triangle area

If you know	Use this
Base and altitude	"Half base times height" method
All 3 sides	Heron's Formula also called Hero's Formula
Two sides and included angle	Side-angle-side method
x, y coordinates of the vertices	Area of a triangle—by formula (Coordinate Geometry) Area of a triangle—box method (Coordinate Geometry)
The triangle is equilateral	Area $\frac{\sqrt{3}}{4}.S^2$

"Half base times height" Method

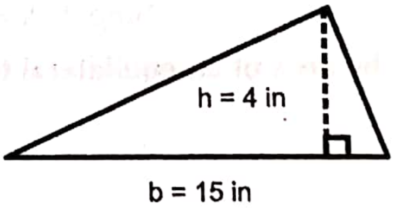
The area of a triangle can be by the formula: $A = \frac{1}{2}.b.h$

Where:

- b is the length of the base
- h is the length of the corresponding altitude

You can choose any side to be the base. It need not be the one drawn at the bottom of the triangle. The altitude must be the one corresponding to the base you choose. The altitude is the line perpendicular to the selected base from the opposite vertex.

Example 1: Find the area of an acute triangle with a base of 15 inches and a height of 4 inches.



Solution:

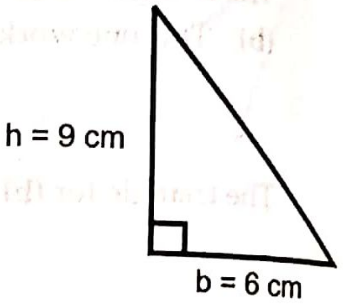
$$A = \frac{1}{2}.b.h$$

$$A = \frac{1}{2}.(15\,\text{in}).(4\,\text{in})$$

$$A = \frac{1}{2}.(60\,\text{in}^2)$$

$A = 30\,in^2$

Example 2: Find the area of a right triangle with a base of 6 centimetres and a height of 9 centimetres.



Solution:

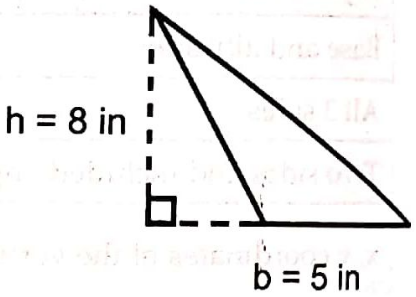
$A = \frac{1}{2}.b.h$

$A = \frac{1}{2}.(6cm).(9cm)$

$A = \frac{1}{2}.(54\,cm^2)$

$A = 27cm^2$

Example 3: Find the area of an obtuse triangle with a base of 5 inches and a height of 8 inches.



Solution:

$A = \frac{1}{2}.b.h$

$A = \frac{1}{2}.b(5in).(8in)$

$A = \frac{1}{2}.(40\,in^2)$

$A = 20\,in^2$

Heron's Formula for the area of a triangle (Hero's Formula)

A method for calculating the area of a triangle when you know the lengths of all three sides. Let a, b, c be the lengths of the sides of a triangle. The area is given by:

- **Step 1:** Calculate "s" (half of the triangles perimeter): $S = \frac{a+b+c}{2}$
- **Step 2:** Then calculate the Area: $A = \sqrt{s(s-a)(s-b)(s-c)}$

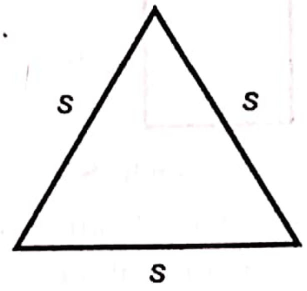
Example 4: What is the area of a triangle where every side is 5 long?

Step 1: $s = (5 + 5 + 5)/2 = 7.5$

Step 2: $A = \sqrt{(7.5 \times 2.5 \times 2.5 \times 2.5)} = \sqrt{(117.1875)} = 10.825.$

The area of an equilateral triangle (all sides congruent) can be found using the formula

$$\text{Area} = \frac{\sqrt{3}}{4} S^2$$



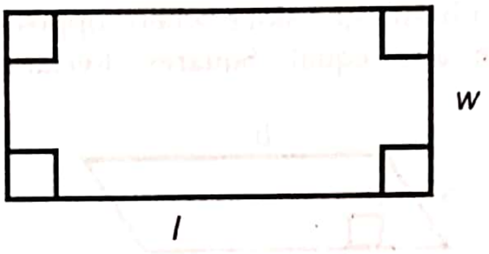
Where s is the length of a side of the triangle.

Note that $\frac{\sqrt{3}}{4}$ is constant that has the value of approximately 0.433, so the formula simplifies a little to Approximate area = $0.433\ S^2$

AREA OF RECTANGLE, SQUARE, PARALLELOGRAM, RHOMBUS AND TRAPEZOID

Rectangle

A four sided flat shape with straight sides where all interior angles are right angles (90°). Opposite side are also parallel and of equal Length.

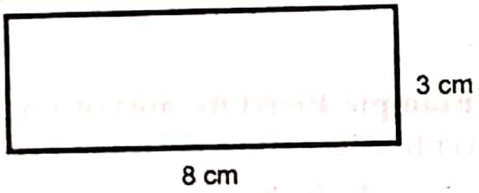


Area: $A = l\,w$.

Example 1: A rectangle has a length of 8 centimeters and a width of 3 centimeters. Find the area.

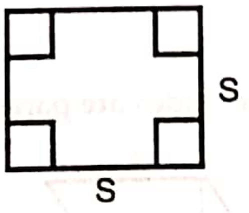
$A = L \,.\, W$

$A = (8\text{ cm}) \cdot (3\text{ cm}) = 24\text{ cm}^2$



Square

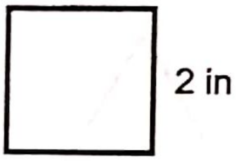
A square is a special type of rectangle which means that it has four equal sides and four equal angles (90-degree angles, or right angles). It can also be defined as a rectangle in which two adjacent sides have equal length.



Area: $A = s \times s = s^2$

Note. Of course, you could call one side l and the other side w and then use the formula $A = l\,w$.

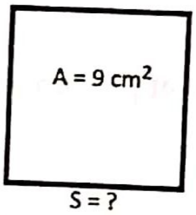
Example 2: Find the area of a square with each side measuring 2 inches.



$$A = S \cdot S$$

$$A = (2\text{ in}) \cdot (2\text{ in}) = 4\text{ in}^2$$

Example 3: The area of a square is 9 square centimeters. How long is one side?



$A = S \cdot S$

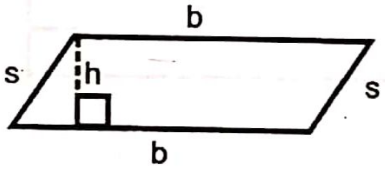
$9\text{ cm}^2 = S^2$

Since $3 \cdot 3 = 9$, we get $3\text{ cm} \cdot 3\text{ cm} = 9\text{ cm}^2$. So S must equal 3 cm.

$S = 3\text{ cm}$

Parallelogram

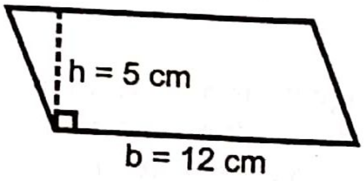
A four sided flat shape with straight sides where opposite sides are parallel, equal in length and opposite angles are also equal. Squares, Rectangles and Rhombuses are all parallelograms.



The height h must make a 90° angle with the base b.

Area: $A = bh$

Example 4: Find the area of a parallelogram with a base of 12 centimeters and a height of 5 centimeters.



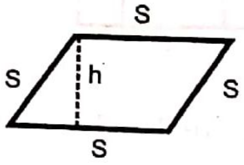
$A = b \cdot h$

$A = (12\text{ cm}) \cdot (5\text{ cm})$

$A = 60\text{ cm}^2$

Rhombus

All sides are of equal length. Opposite sides are parallel and opposite angles are equal.

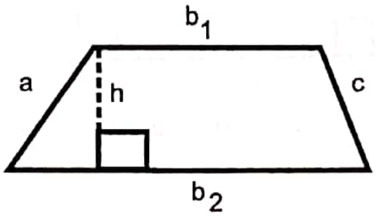


The height h must make a $90°$ angle with the base b.

Area: $A = sh$.

Trapezoid

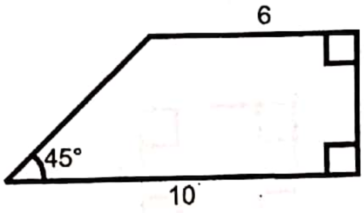
A four sided flat shape with straight sides and a pair of opposite sides is parallel. The sides that are parallel are called bases. The other sides are legs which may or may not be parallel. It is also called Trapezium. The sides *a* and *c* can be of equal length, but it is not mandatory. The top and bottom b_1 and b_2 are parallel.



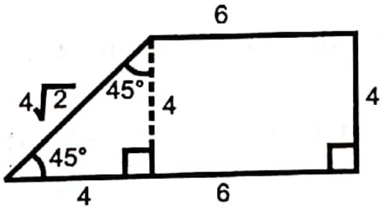
The height ***h*** must make a 90° angle with the base ***b***.

$$\text{Area: } A = = \frac{b^1 + b^2}{2}(h)$$

Example 5: Find the area of the trapezoid given below.



The bases of the trapezoid are the parallel sides, and we're told their lengths are 6 and 10. but to find the area, we also need to find the trapezoid's height, which isn't given. To do that, split the trapezoid into a rectangle and a 45-45-90 triangle by drawing in the height.



Once you've drawn in the height, you can split the base that's equal to 10 into two parts: The base of the rectangle is 6, and the leg of the triangle is 4. Since the triangle is 45-45-90, the two legs must be equal. This leg, though, is also the height of the trapezoid. So the height of the trapezoid is 4.

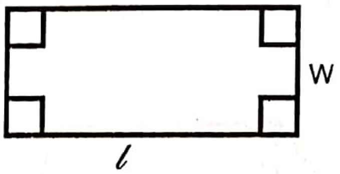
Now you can plug the numbers into the formula: $Area = \frac{b_1 + b_2}{2}(h)$

$$\text{Area} = \frac{6+10}{2}(4) = 8(4) = 32$$

PERIMETER

Perimeter is the distance around a two dimensional shape, a measurement of the distance around something; the length of the boundary. The word comes from the Greek *peri* (around) and *meter* (measure). We will discuss the perimeters and areas of the rectangle, square, parallelogram, rhombus, trapezoid, triangle, and circle.

Rectangle

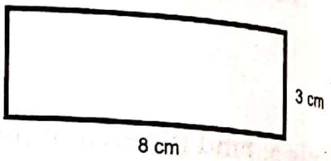


Perimeter: $P = l + l + w + w$

$P = 2l + 2w.$

There is an implied multiplication between the number 2 and the letters!

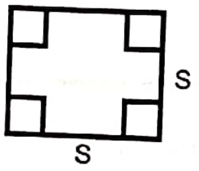
Example 1: A rectangle has a length of 8 centimeters and a width of 3 centimeters. Find the perimeter.



Solution 1: P = 8 cm + 8cm + 3 cm + 3 cm = 22 cm

Solution 2: P = 2(8 cm) + 2(3 cm) = 16 cm + 6 cm = 22 cm

Square

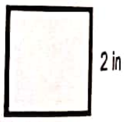


Perimeter: $P = s + s + s + s$

$P = 4s.$

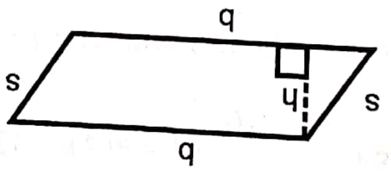
Note. Of course, you could call one side l and the other side w and then use the formula $P = 2l + 2w$.

Example 2 : Find the perimeter of a square with each side measuring 2 inches.



Solution: P = 2 in + 2 in + 2 in + 2 in = 8 in

Parallelogram



Perimeter: $P = s + s + b + b$

$P = 2s + 2b.$

There is an implied multiplication between the number 2 and the letters!

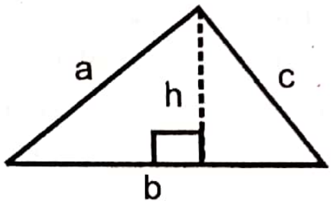
Example 3: Find the perimeter of a parallelogram that has sides of 30 ft and 18 ft.

Perimeter Formula: $P = 2s + 2b$

$$P = 2(30) + 2(18) = 60 + 36 = 96$$

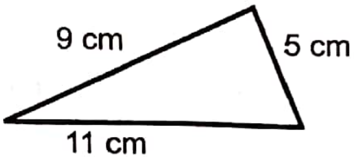
The perimeter of the parallelogram is 96 ft.

Triangle



Perimeter: $P = a + b + c$

Example 4: Find the perimeter of a triangle with sides measuring 5 centimetres, 9 centimetres and 11 centimetres.



P = 5 cm + 9 cm + 11 cm = 25 cm

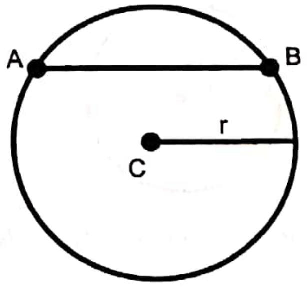
CIRCLE

A circle is the collection of points equidistant from a given point, called the center. Circles are named after their center points. All circles contain 360°.

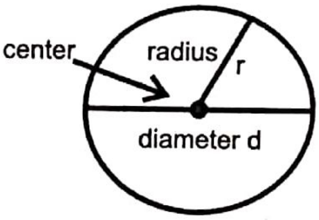
The distance from the center to any point on the circle is called the **radius** (r). Radius is the most important measurement in a circle, because if you know a circle's radius, you can figure out all its other characteristics, such as its *area*, *diameter*, and *circumference*.

The **diameter (d)** of a circle stretches between endpoints on the circle, passing through the center. A *chord* also extends from endpoint to endpoint on the circle, but it does not necessarily pass through the center. In the following figure, point C is the center of the circle, *r* is the radius, and AB is a chord. The diameter is twice as long as the radius.

Diameter = d = r + r = 2r



The perimeter of a circle is called the **circumference**. The circle is a geometric figure with a special relationship between its distance around the circle (circumference) and its diameter. If we divide the circumference of any circle by its diameter, the quotient is always the same number, namely the number π (pi).



π (Pronounced **pi**) is represented by the Greek letter π and its value is a non-repeating, non-terminating decimal approximately equal to ***3.141592654*** or better known as ***3.14***.

Both the circumference and the area of a circle use the number π in its formulas.

Circumference (perimeter or distance around the circle): $C = 2\pi r$ or $C = \pi . d$

Area: $A = \pi r^2$

Example 1: The radius of a circle is 2 inches. What is the diameter?

Solution: $d = 2r$

$d = 2 \cdot (2 \text{ in})$

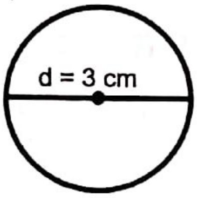
$d = 4 \text{ in}$

Example 2: The diameter of a circle is 3 centimeters. What is the circumference?

Solution: $C = \pi d$

$C = 3.14\,(3 \text{ cm})$

$C = 9.42 \text{ cm}$



Example 3: The radius of a circle is 2 inches. What is the circumference?

Solution: $d = 2r$

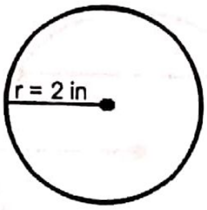
$d = 2 \cdot (2 \text{ in})$

$d = 4 \text{ in}$

$C = \pi d$

$C = 3.14 \cdot (4 \text{ in})$

$C = 12.56 \text{ in}$



Example 4: The circumference of a circle is 15.7 centimeters. What is the diameter?

Solution: $C = \pi d$

$15.7 \text{ cm} = 3.14\, d$

$15.7 \text{ cm} \div 3.14 = d$

$d = 15.7 \text{ cm} \div 3.14$

$d = 5 \text{ cm}$

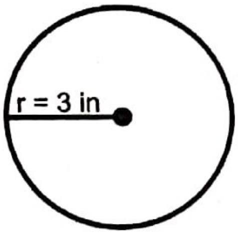
Example 5: The radius of a circle is 3 inches. What is the area?

Solution: $A = \pi r r = \pi r^2$

$A = 3.14 \cdot (3\ in) \cdot (3\ in)$

$A = 3.14 \cdot (9\ in)$

$A = 28.26\ in$



Example 6: The diameter of a circle is 8 centimeters. What is the area?

Solution: $d = 2\ r$

$8\ cm = 2\ r$

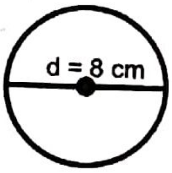
$8\ cm \div 2 = r$

$r = 4\ cm$

$A = \pi r r$

$A = 3.14 \cdot (4\ cm) \cdot (4\ cm)$

$A = 50.24\ cm$



Example 7: The area of a circle is 78.5 square meters. What is the radius?

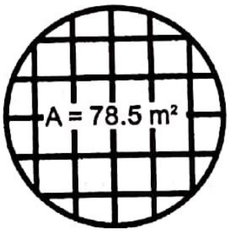
Solution: $A = \pi r r$

$78.5\ m = 3.14 \cdot r\ \ r$

$78.5\ m \div 3.14 = r\ \ r$

$25\ m = r\ r$

$r = 5\ m$



EXERCISE

Problem 1: Find the area of a rectangle whose length is 14 ft and whose width is 9 ft.

Area Formula: $A = l\, w$

$A = 14(9) = 126$

The area of the rectangle is 126 ft².

Problem 2: Find the area of a square that measures 5 cm on a side.

Area Formula: $A = s \times s = s^2$

$$A = 5 \times 5 = 25$$

The area of the square is 25 cm².

Problem 3: Find the area of a parallelogram with base 30 ft and height is 18 ft.

Area Formula: $A = bh$

$$\mathrm{A} = 30(18) = 540$$

The area of the parallelogram is 540 ft².

Problem 4: Find the area of a triangle with base 12 inches and height 6 inches.

Area: $A = \frac{1}{2}bh$

$$A = \frac{1}{2}12(6) = 36$$

The area of the triangle is 36 in².

Problem 5: A garden plot turned out to be a parallelogram with base 12 ft and height 6ft. What is the area of the garden plot?

Area Formula: $A = bh$, where h is the height and b is the base

$$\mathrm{A} = 12(6) = 72$$

The area of the garden plot is 72 ft².

Problem 6: A parking lot plot turned out to be a triangle with base 6.2 m and height 10 m. What is the area of the parking lot?

Area Formula: $A = \frac{1}{2}bh$ where h is the height and b is the base

$$A = \frac{1}{2}(6.2)(10) = 31m^2$$

The area of the parking lot is 31 m².

Problem 7: Find the diameter of a circle whose circumference is 36 metres rounded to the nearest hundredth.

We know the formula of the circumference to be $C = 2\pi r$ and we know that the diameter $d = 2r$. We are given $C = 36$. Therefore, $36 = 2\pi r$, Then

$$d = \frac{36}{\pi}$$

$$d \approx 11.46$$

The diameter of a circle whose circumference is 36 metres is approximately 11.46 metres.

Problem 8: Find the area of a circle whose diameter is 12 cm. Round the answer to a whole number.

Diameter Formula: $d = 2r$

Area Formula: $A = \pi r^2$

To find the area we need a radius. Given is the diameter, which is twice the radius. Therefore, the radius must be 6 cm.

$$\text{Then } A = \pi(6)^2 = \pi(36) = 113.0973355...$$

The area of the circle is approximately 113 cm².

Problem 9: Find the circumference of a circle whose diameter is 16 cm. Round the answer to a whole number.

Diameter Formula: $d = 2r$

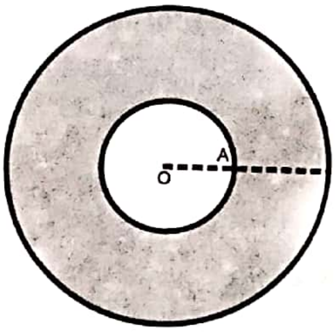
Circumference formula: $C = 2\pi r$

To find the circumference we need a radius. Given is the diameter, which is twice the radius. Therefore, the radius must be 8 cm.

$$\text{Then } C = 2\pi(8) = 50.26548246...$$

The circumference of the circle is approximately 50 cm.

Problem 10: Find the area of the shaded ring rounded to the nearest hundredth. Assume that the point O is the centre of both the small and the large circle. The distance OA, which is the radius of the small circle, is 5 meters and the distance OB, which is the radius of the large circle, is 11 metres. Round the answer to a whole number.



We can find the area of the shaded ring by subtracting the area of the small circle from the area of the large circle!

That is,

$$A_{shaded} = A_{large} - A_{small}$$

and given the area formula $A = \pi r^2$, we get

$A_{shaded} = \pi(11^2) - \pi(5^2)$

$A_{shaded} = \pi(121) - \pi(25)$

$A_{shaded} = 301.5928947...$

The area of the shaded ring measures approximately 302 m².

Problem 11: The measure of one supplementary angle is twice the measure of the second. What is the measure of each angle?

Let x be the measure of the first angle. Then, the second angle is 2x

Since the angle are supplementary, they add up to 180°

$x + 2x = 180°$

$3x = 180°$

Since $3 \times 60 = 180$, $x = 60$

The measure of the first angle is 60°

The measure of the second is $2x = 2 \times 60 = 120°$

Problem 12: A right triangle has acute angles whose measures are in the ratio 1:3. Find the measure of these acute angles.

Thing to know: The sum of the angles in a triangle is equal to 180°

Meaning of the ratio 1:3

This means that the second acute angle is 3 times bigger than the first acute angle

Let x be the first acute angle, then the second acute angle will be 3x

$x + 3x + 90° = 180°$

$4x + 90° = 180°$

$4x + 90° - 90° = 180° - 90°$

$4x = 90°$

Since $4 \times 22.5 = 90°$, $x = 22.5°$

The second angle is $3x = 3 \times 22.5 = 67.5$

The measure of the two acute angles are 22.5 and 67.5

Problem 13: The sum of the measures of the angles of an n-gon is 2340°. How many sides does this n-gon have?

To solve this problem, you need to know the following formula:

Sum of the angles in an n-gon = $(n - 2)\times 180°$

n is the number of sides. So just plug in the numbers and solve

$2340° = (n - 2)\times 180°$

$2340° = 180°n - 360°$

$2340° + 360° = 180°n - 360° + 360°$

$2700° = 180°n$

Divide both sides 180°

$(2700° \div 180°) = (180° \div 180°)n$

$15 = n$

The n-gon has 15 sides

Problem 14: The diameter of a penny is 0.750 inch and the diameter of a quarter is 0.955 inch. You put the penny on top and exactly in the middle of the quarter. Since the coin is smaller, it will not cover completely the quarter. What is the area of the portion that is not covered? Will the area change if the coin is not centered?

We can use A = πr2 since the coin is shaped like a circle

Let B stand for the area of the portion not covered

B = area of quarter - area of penny

r = 0.375 inch for the penny and r = 0.4775 for the quarter

$B = 3.14 \times 0.4775 \times 0.4775 - 3.14 \times 0.375 \times 0.375$

$B = 0.715 - 0.441$

B = 0.274 inch2

As long as the coin remains inside the quarter, the area should stay the same

Problem 15: If one side of a square is doubled in length and the adjacent side is decreased by two centimeters, the area of the resulting rectangle is 96 square centimeters larger than that of the original square. Find the dimensions of the rectangle.

square's side length: x

one side is doubled: 2x

next side is decreased by two: x – 2

square's area: x2

rectangle's area: (2x)(x – 2) = 2x2 – 4x

new area is 96 more than old area: 2x2 – 4x = x2 + 96

$$2x2 - 4x = x2 + 96$$

$$x2 - 4x - 96 = 0$$

$$(x - 12)(x + 8) = 0$$

$$x = 12 \text{ or } x = -8$$

Length of rectangle cannot be negative or less than zero therefor, "x = 12" (the only reasonable solution for the square) means that the sides of the rectangle have lengths 2(12) and (12) – 2:

Problem 16. A pole is broken such that top of the pole touches the ground. If pole is broken at a height of 72 feet from the ground and top of broken pole touches the ground at a distance of 54 feet from its base, find the length of pole.

a. 163 feet b. 158 feet c. 160 feet d. 162 feet

Problem 17 A triangle has three angles, which we call the first, second and third angle here. The second angle is twice the first angle. The third angle is 51°. What is the value of the first angle?

Problem 18 In a triangle, one angle is a complement of the other. What is the value of the third angle?

Problem 19 Exterior angle of a triangle is 40°, and ratio of opposite interior angles is 7:3. Find smaller of two interior angles.

Problem 20: Momina is walking in south direction. After walking for 27 meters, she takes a right turn, and walks for another 36 meters. She is at a distance of __________ meters from starting point.

Answers: Problem 16 to 20

Problem 11	162 ft
Problem 12	43°
Problem 13	90°
Problem 14	12°
Problem 15	45m

MULTIPLE CHOICE QUESTIONS (MCQS)

1. **If two angles are said to be supplementary angles and one of angle is of 122 ° then other angle is of**

   (a) 35° (b) 58° (c) 60° (d) 32°

2. **Angle which is less than 360° and larger than 180° is classified as**

   (a) Acute angle (b) Obtuse angle (c) Right angle (d) Reflex angle

3. **If line segment is extended in two directions indefinitely from each of two points then it is classified as**

   (a) Intersecting line (b) Plane (c) Line (d) Ray

4. **Flat surface in which two points are joined by using straight line is classified as**

   (a) Line (b) Ray (c) Intersecting line *(d)* Plane

5. **Out of following options, two angles that are together classified as complementary angles are**

   (a) 120° and 60° (b) 50° and 30°

   (c) 65° and 25° (d) 70° and 30°

6. **On a line, sum of adjacent angles is equal to**

   (a) 90° *(b)* 180° (c) 140° (d) 120°

7. **A rectangular field is 40m long and 30m wide. perimeter of rectangular field is**

(a) 200m (b) 180m (c) 160m (d) 140m

8. **Type of quadrilateral which has one pair of parallel sides is called**

(a) Triangle (b) Semi-circle (c) Parallelogram (d) Trapezium

9. **Kind of triangle which has two equal sides is**

(a) Exterior angle (b) Equilateral triangle

(c) Isosceles triangle (d) Scalene triangle

10. **Right-angled triangle must have**

(a) Four right angles (b) Three right angles

(c) One right angle (d) Two right angles

CHAPTER 25

MEASURES OF CENTRAL TENDENCY

A measure of central tendency (also referred to as measures of center or central location) is a summary measure that attempts to describe a whole set of data with a single value that represents the middle or center of its distribution. There are three main measures of central tendency: **mode, median** and **mean.** Each of these measures describes a different indication of the typical or central value in the distribution.

MODE

The mode is the most commonly occurring value in a data.

Example: Consider this dataset showing the retirement age of 11 people:

54, 54, 54, 55, 56, 57, 57, 58, 58, 60, 60

This table shows a simple frequency distribution of the retirement age data.

AGE	FREQUENCY
54	3
55	1
56	1
57	2
58	2
60	2

The most commonly occurring value is 54, therefore the mode of this distribution is 54 years.

Finding the Mode

Example: A football team scored 2, 4, 4, 5, 5, 5 goals in six consecutive matches. Find the mode

Write down all of the numbers in the set. In this case, you're working with the numbers 2, 4, 5, 5, 4, and 5. It can help to order them from least to greatest here too.

2, 4, 4, 5, 5, 5

Find the number that occurs the most. Think: "Mode is most." In this example, the number 5 occurs the most, so that is the mode. If there are two numbers that occur the most, then that set is "**bimodal**," and if more than two numbers occur the most, then the set is "**multi-modal**."

If a distribution is termed to be multimodal, there are at least two modes in the distribution. This means that at least two numbers appear the greatest amount of times in the distribution. For example, if you have a set of data in which three numbers each appear five times and no other numbers appear more than five times, then you have a multimodal distribution with three modes.

Example: How to calculate the mode in the following dataset? 1, 2, 3, 4, 1, 2, 1, 2, 1, 2

Start by lining up the data in ascending order, so it's organized in an easily readable fashion:

1, 1, 1, 1, 2, 2, 2, 2, 3, 4.

Then, figure out which number appears most often. This is the mode.

In the dataset, there are actually two modes -- 1 and 2.

They both appear 4 times and nothing else appears more.

EXERCISE

Question 1: The following is the number of problems that Ms. Erum assigned for homework on 10 different days. What is the mode?

8, 11, 9, 14, 9, 15, 18, 6, 9, 10

Solution: Ordering the data from least to greatest, we get:

6, 8, 9, 9, 9, 10, 11 14, 15, 18

The mode is 9.

Question 2: In a crash test, 11 cars were tested to determine what impact speed was required to obtain minimal bumper damage. Find the mode of the speeds given in miles per hour below.

24, 15, 18, 20, 18, 22, 24, 26, 18, 26, 24

Solution: Ordering the data from least to greatest, we get:

15, 18, 18, 18, 20, 22, 24, 24, 24, 26, 26

Since both 18 and 24 occur three times, the modes are 18 and 24 miles per hour. This data set is bimodal.

Question 3: A marathon race was completed by 5 participants. What is the mode of these times given in hours?

2.7 hr, 8.3 hr, 3.5 hr, 5.1 hr, 4.9 hr

Solution: Ordering the data from least to greatest, we get:

2.7, 3.5, 4.9, 5.1, 8.3

Answer: Since each value occurs only once in the data set, there is no mode for this set of data.

Question 4: On a cold winter day in January, the temperature for 9 North American cities is recorded in Fahrenheit. What is the mode of these temperatures?

-8, 0, -3, 4, 12, 0, 5, -1, 0

Solution: Ordering the data from least to greatest, we get:

-8, -3, -1, 0, 0, 0, 4, 5, 12

The mode of these temperatures is 0.

Let's compare the results of the last two questions. In questions 3, each value occurs only once, so there is no mode. In Example 4, the mode is 0, since 0 occurs most often in the set. Do not confuse a mode of 0 with no mode.

Advantage of the mode: The mode has an advantage over the median and the mean as it can be found for both numerical and categorical (non-numerical) data.

Limitations of the mode: The are some limitations to using the mode. In some distributions, the mode may not reflect the centre of the distribution very well. When the distribution of retirement age is ordered from lowest to highest value, it is easy to see that the centre of the distribution is 57 years, but the mode is lower, at 54 years.

54, 54, 54, 55, 56, 57, 57, 58, 58, 60, 60

It is also possible for there to be more than one mode for the same distribution of data, (bi-modal, or multi-modal). The presence of more than one mode can limit the ability of the mode in describing the centre or typical value of the distribution because a single value to describe the centre cannot be identified.

In some cases, particularly where the data are continuous, the distribution may have no mode at all (i.e. if all values are different).

In cases such as these, it may be better to consider using the median or mean, or group the data in to appropriate intervals, and find the modal class.

MEDIAN

The median is the middle value in distribution when the values are arranged in ascending or descending order. The median divides the distribution in half (there are 50% of observations on either side of the median value). In a distribution with an odd number of observations, the median value is the middle value.

Looking at the retirement age distribution (which has 11 observations), the median is the middle value, which is 57 years:

54, 54, 54, 55, 56, 57, 57, 58, 58, 60, 60

When the distribution has an even number of observations, the median value is the mean of the two middle values. In the following distribution, the two middle values are 56 and 57, therefore the median equals 56.5 years as we get the average of the two values.

52, 54, 54, 54, 55, 56, 57, 57, 58, 58, 60, 60

How to find Median

First Step: Arrange the observations in an ordered array.

Second Step: For an array with an **odd number** of terms, the median is the middle number.

Third Step: For an array with an **even number** of terms, the median is the average of the two middle numbers.

Locating the Median: The median's location in an ordered array is found by formula (n+1)/2

Example: Find the median number of tracks on Honey Singh CDs.

No of tracks were:

9 13 9 11 9 13 11 9 10 8 11

Put the numbers in numerical order:

8 9 9 9 9 10 11 11 11 13 13

Find the middle number:

$$\frac{n+1}{2} = \frac{11+1}{2} = \frac{12}{2} = 6th\,Value$$

$$\underline{8\ 9\ 9\ 9\ 9}\ \boxed{10}\ \underline{11\ 11\ 11\ 13\ 13}$$

The middle number is 10, so the median is 10.

Finding the median with an even number of values

To find the median of the numbers: 5 11 12 4 8 21.

Put the results in order:

4 5 8 11 12 21.

Find the middle number or numbers:

$$\frac{n+1}{2} = \frac{6+1}{2} = \frac{7}{2} = 3.5th\,Value$$

We need to find mean of 3rd and 4th value.

$$4\ \ 5\ \ \underline{8\ \ 11}\ \ 12\ \ 21.$$

The median is therefore average of the 3rd and 4th value:

$$(8+11) \div 2 = 9.5$$

Advantage of the median: The median is less affected by outliers and skewed data than the mean, and is usually the preferred measure of central tendency when the distribution is not symmetrical.

Limitation of the median: The median cannot be identified for categorical nominal data, as it cannot be logically ordered.

EXERCISE

Question 1: The Obama family drove through 7 states on their summer vacation. Gasoline prices varied from state to state. What is the median gasoline price?

$1.79, $1.61, $1.96, $2.09, $1.84, $1,75, $2.11

Solution: Ordering the data from least to greatest, we get:

$1.61, $1.75, $1.79, **$1.84,** $1.96, $2.09, $2.11

The median gasoline price is $1.84.

Question 2: During the first marking period, Imran's math quiz scores were 90, 92, 93, 88, 95, 88, 97, 87, and 98. What was the median quiz score?

Solution: Ordering the data from least to greatest, we get:

87, 88, 88, 90, **92,** 93, 95, 97, 98

The median quiz score was 92.

In each of the questions above, there is an odd number of items in each data set. In question 1, there are 7 numbers in the data set; in question 2 there are 9 numbers. Let's look at some question in which there is an even number of items in the data set.

Question 3: A marathon race was completed by 4 participants. What was the median race time?

2.7 hr, 8.3 hr, 3.5 hr, 5.1 hr

Solution: Ordering the data from least to greatest, we get:

2.7, 3.5, 5.1, 8.3

Since there is an even number of items in the data set, we compute the median by taking the mean of the two middle numbers.

2.7, **3.5, 5.1,** 8.3

$$3.5 + 5.1 = 8.6$$

$$\begin{array}{r} 4.3 \\ 2\overline{)8.6} \end{array}$$

The median race time was 4.3 hr.

Question 4: The salaries of 8 employees who work for a small company are listed below. What is the median salary?

$40,000, $29,000, $35,500, $31,000, $43,000, $30,000, $27,000, $32,000

Solution: Ordering the data from least to greatest, we get:

$27,000, $29,000, $30,000, $31,000, $32,000, $35,500, $40,000, $43,000

Since there is an even number of items in the data set, we compute the median by taking the mean of the two middlemost numbers.

$27,000, $29,000, $30,000, **$31,000, $32,000,** $35,500, $40,000, $43,000

$$\$31{,}000 + \$32{,}000 = \$63{,}000$$

$$\begin{array}{r} \$31{,}500 \\ 2\overline{)\$63{,}000} \end{array}$$

The median salary is $31,500.

MEAN OR AVERAGE

The mean is the sum of the value of each observation in a dataset divided by the number of observations. This is also known as the arithmetic average.

Looking at the retirement age distribution again:

54, 54, 54, 55, 56, 57, 57, 58, 58, 60, 60

The mean is calculated by adding together all the values

54+54+54+55+56+57+57+58+58+60+60 = 623

and dividing by the number of observations (11) which equals 56.6 years.

So, if we have n values in a data set and they have values x_1, x_2, ..., x_n, the sample mean, usually denoted by $\bar{x}$ (pronounced x bar), is:

$$\bar{x} = \frac{(x_1 + x_2 + \ldots + x_n)}{n}$$

This formula is usually written in a slightly different manner using the Greek capital letter, $\sum$, pronounced "sigma", which means "sum of":

$$\text{Sample Mean} = \bar{x} = \frac{\sum x}{n}$$

You may have noticed that the above formula refers to the sample mean. So, why have we called it a sample mean? This is because, in statistics, samples and populations have very different meanings and these differences are very important, even if, in the case of the mean, they are calculated in the same way. To acknowledge that we are calculating the population mean and not the sample mean, we use the Greek lower case letter "mu", denoted as μ:

$$\text{Population Mean} = \mu = \frac{\sum x}{N}$$

EXERCISE

Question 1: Find the mean driving speed for 6 different cars on the same highway.

66 mph, 57 mph, 71 mph, 54 mph, 69 mph, 58 mph

Solution: 66 + 57 + 71 + 54 + 69 + 58 = 375

$$\begin{array}{r} 62.5 \\ 6\overline{)375.0} \end{array}$$

The mean driving speed is 62.5 mph.

Question 2: The Butt family drove through 4 cities on their summer vacation. Gasoline prices varied from city to city \$1.79, \$1.61, \$1.96, \$2.08 respectivel. What is the mean gasoline price?

Solution: \$1.79 + \$1.61 + \$1.96 + \$2.08 = \$7.44

$$\begin{array}{r} \$1.86 \\ 4\overline{)\$7.44} \end{array}$$

Answer: The mean gasoline price is $1.86.

Question 3: A marathon race was completed by 5 participants in the times given below. What is the mean race time for this marathon?

2.7 hr, 8.3 hr, 3.5 hr, 5.1 hr, 4.9 hr

Solution: $2.7 + 8.3 + 3.5 + 5.1 + 4.9 = 24.5$

$$\begin{array}{r} 4.9 \\ 5\overline{)24.5} \end{array}$$

The mean race time is 4.9 hr

Question 4: Find the mean swimming time rounded to the nearest tenth:

2.6 min, 7.2 min, 3.5 min, 9.8 min, 2.5 min

Solution: $2.6 + 7.2 + 3.5 + 9.8 + 2.5 = 25.6$

$$\begin{array}{r} 5.12 \\ 5\overline{)25.60} \end{array}$$

The mean swimming time to the nearest tenth is 5.1 min.

Advantage of the mean: The mean can be used for both continuous and discrete numeric data.

Limitations of the mean: The mean cannot be calculated for categorical data, as the values cannot be summed.

As the mean includes every value in the distribution the mean is influenced by outliers and skewed distributions.

Additional Examples

Example 1. Find the average of all prime numbers between 30 and 50.

Solution: There are five prime numbers between 30 and 50.

They are 31, 37, 41, 43 and 47.

$$\therefore \quad \text{Required average} = \frac{(31+37+41+43+47)}{5} = \frac{199}{5} = 39.8$$

Example 2. A batsman makes a score of 87 runs in the 17th inning and thus increases his average by 3. Find his average after 17th inning.

Solution: Let the average after 17th inning = x

Then, average after 16th inning = $(x-3)$

$\therefore \quad 16(x-3) + 87 = 17x$ or $x = (87-48) = 39$

Example 3. A man purchased 5 toys at the rate of Rs. 200 each, 6 toys at the rate of Rs. 250 each and 9 toys at the rate of Rs. 300 each. Calculate the average cost of one toy.

Solution: Price of 5 toys = 200×5 = Rs. 1000

Price of 6 toys = 250×6 = Rs. 1500

Price of 9 toys = 300 x 9 = Rs. 2700

Total number of toys = 5 + 6 + 9 = 20

Average price of 1 toy = $\frac{1000+1500+2700}{20} = \frac{5200}{20}$ = Rs.260

Example 4. Find the average of first 81 natural numbers.

Rule 1 (Formulas for such questions):

(a) The average of first n natural numbers is $\frac{n+1}{2}$

(b) The average of square of natural numbers till n is $\frac{(n+1)(2n+1)}{6}$

(c) The average of cubes of natural numbers till n is $\frac{n(n+1)^2}{4}$

(d) The average of odd numbers from 1 to n is $\frac{\text{last odd number} + 1}{2}$

(e) The average of even numbers from 1 to n is $\frac{\text{last even number} + 2}{2}$

Solution: The required average $= \frac{n+1}{2} = \frac{81+1}{2} = 41$

Example 5. What is the average of squares of the natural numbers from 1 to 41?

Solution: The required average $= \frac{(n+1)(2n+1)}{6} = \frac{(41+1)(2\times 41+1)}{6} = \frac{42\times 83}{6} = \frac{3486}{6} = 581$

Example 6. What is the average of odd numbers from 1 to 40?

Solution: The required average $= \frac{\text{last odd number} + 1}{2} = \frac{39+1}{2} = 20$

Example 7. What is the average of even numbers from 1 to 81?

Solution: The required average $= \frac{\text{last even number} + 2}{2} = \frac{80+2}{2} = 41$

Rule 2

(a) If n is odd: The average of n consecutive numbers, consecutive even numbers or consecutive odd numbers is always the middle number.

(b) If n is even: The average of n consecutive numbers, consecutive even numbers or consecutive odd numbers is always the average of the middle two numbers.

Example 8. Find the average of 7 consecutive numbers 3, 4, 5, 6, 7, 8, 9.

Solution: The required average = middle number = 6

Example 9. Find the average of consecutive odd numbers 21, 23, 25, 27, 29, 31, 33, 35.

Solution: The required average

= average of middle two numbers

$$= \text{average of 27 and 29} = \frac{27+29}{2} = 28$$

Example 10. Find the mean, median and mode for the following list of values:

13, 18, 13, 14, 13, 16, 14, 21, 13

The mean is the usual average, so:

$$(13 + 18 + 13 + 14 + 13 + 16 + 14 + 21 + 13) \div 9 = 15$$

The median is the middle value, so I'll have to rewrite the list in order:

13, 13, 13, 13, 14, 14, 16, 18, 21

There are nine numbers in the list, so the middle one will be the $(9 + 1) \div 2 = 10 \div 2 =$ 5th number:

13, 13, 13, 13, **14**, 14, 16, 18, 21

So the median is 14.

The mode is the number that is repeated more often than any other, so 13 is the mode.

Mean: 15, Median: 14, Mode: 13

MULTIPLE CHOICE QUESTIONS

Find out the correct answer.

Q. 1. The average of 7 consecutive numbers is 33. The largest of these numbers is:

(a) 36 **(b)** 33 **(c)** 30 **(d)** 28

Ans. (a)

Solution: Let the numbers be x, x+1, x+2, x+3, x+4, x+5 and x+6

$$\text{Then, } \frac{x+x+1+x+2+x+3+x+4+x+5+x+6}{7} = 33$$

$$\text{or } \frac{7x+21}{7} = 33 \text{ or } \frac{7(x+3)}{7} = 33 \text{ or } x+3 = 33 \text{ or } x = 30$$

$\therefore$ Largest number $= x + 6 = (30 + 6) = 36$

Q. 2. The average height of 30 boys out of a class of 50, is 160 cm. If the average height of the remaining boys is 165 cm, the average height of the whole class (in cm) is:

(a) 161 **(b)** 162 **(c)** 163 **(d)** 164

Ans. (b)

Solution: Required average $= \left(\frac{30 \times 160 + 20 \times 165}{50}\right) = 162 \text{ cm}$

Q. 3. The average of three numbers is 20. If two numbers are 16 and 22, the third is:

(a) 22 **(b)** 20 **(c)** 19 **(d)** 18

Ans. (a)

Solution: Total sum of three numbers $= (20 \times 3) = 60$

Third number $= 60 - (16+22) = 22$

Q. 4. **The average age of 30 students of a class is 12 years. The average age of a group of 5 of the students is 10 years and that of another group of 5 of them is 14 years. What is the average age of the remaining students?**

(a) 8 years **(b)** 10 years **(c)** 12 years **(d)** 14 years

Ans. **(c)**

Solution: Total age of 20 students = $(30 \times 12) - (5 \times 10 + 5 \times 14) = 240$ years

$\therefore$ Required average = $\left(\frac{240}{20}\right) = 12$ years

Q. 5. **The mean of 100 observations was calculated as 40. It was found later on that one of the observations was misread as 83 instead of 53. The correct mean is:**

(a) 39 **(b)** 39.7 **(c)** 40.3 **(d)** 42.7

Ans. **(b)**

Solution: Correct sum = $(40 \times 100 + 53 - 83) = 3970$

$\therefore$ Correct mean = $\frac{3970}{100} = 39.7$

Q. 6. **A motorist travels to a place 150 km away at an average speed of 50 km per hour and returns at 30 km per hour. His average speed for the whole journey in km per hour is:**

(a) 35 **(b)** 37 **(c)** 37.5 **(d)** 40

Ans. **(c)**

Solution: Average speed = $\frac{2xy}{x+y}$ km/hr = $\left(\frac{2\times50\times30}{50+30}\right)$ km/hr = 37.5 km/hr

Q. 7. **A cricketer has a certain average for 9 innings. In the tenth inning, he scores 100 runs, thereby increasing his average by 8 runs. His new average is:**

(a) 20 runs **(b)** 24 runs **(c)** 28 runs **(d)** 32runs

Ans. **(c)**

Solution: Let average for 9 innings be x. Then,

$$\frac{9x+100}{10} = x+8 \Rightarrow 10x + 80 = 9x + 100 \Rightarrow x = 20$$

$\therefore$ New average = $(x + 8) = 28$ runs

Q. 8. **Five years ago, the average age of P and Q was 15 years. Average age of P, Q and R today is 20 years. How old will R be after 10 years?**

(a) 35 years **(b)** 40 years **(c)** 30 years **(d)** 50 years

Ans. **(c)**

Solution: (P + Q), five years ago = $(15 \times 2) = 30$ years

(P + Q), now = $(30 + 5 \times 2)$ years = 40 years

(P + Q + R), now = (20×3) years = 60 years

$\therefore$ R, now = $(60 - 40)$ years = 20 years

R, after 10 years = $(20 + 10)$ years = 30 years

Q. 9. **The daily earnings of a taxi driver during a week are: Rs. 60, Rs. 65, Rs. 70, Rs. 52.50, Rs. 63, Rs. 73 and Rs. 68. What is his average daily earning for the week?**

(a) Rs. 74.50 **(b)** Rs. 54.50 **(c)** Rs. 64.50 **(d)** Rs. 84.50

Ans. **(c)**

Solution: Total earning for 7 days

= Rs. (60 + 65 + 70 + 52.50 + 63 + 73 + 68)

= Rs. 451.50

Average daily earning = Rs. $\frac{451.50}{7}$ = Rs. 64.50

Q. 10. In a class of 20 students in an examination in Mathematics 2 students scored 100 marks each, 3 got zero each and the average of the rest was 40. What is the average of the whole class?

(a) 40 marks **(b)** 35 marks **(c)** 32 marks **(d)** 45 marks

Ans. (a)

Solution: Marks scored by 2 students = 100 × 2 = 200

Marks scored by 3 students = 3 × 0 = 0

Marks scored by 15 students = 15 × 40 = 600

∴ Marks scored by 20 students = 200 + 0 + 600 = 800

∴ Average marks = $\frac{800}{20}$ = 40

SELF-EXERCISE

1. Ali travels a distance of 300 km in 5 hours. Find the average speed of Ali.
2. Sana covers a distance of 244 km in 4 hours. Find the average speed of Sana.
3. How many miles does a bus travels in 5.5 hours, if its average speed is 40 miles/h?
4. How many miles does a train travels in 7 hours, if its average speed is 80 miles/h?
5. Calculate the total time taken by a bus to cover a distance of 225 km at an average speed 45 km/h.
6. Calculate the total time taken by a train to cover a distance of 320 km at an average speed 40 km/h.
7. Fahad travels for 5 hours at 40 mile/h. Find the total distance traveled.
8. Fahad travels for 3 hours at 40 mile/h and then covers a distance of 95 miles in next 2 hours. Find the average.
9. Aslam travels at the speed of 65 km/h for 4 hours, then reaches at the motorway and travels at the speed of 80 km/h for 2 hours. What will be average speed of his car?
10. Fatima drives a distance of 300 km at an average speed of 40 km/h, and then returns at an average speed of 60 km/h. What will be her average speed for the entire trip?
11. If the average of 2, 3, 4, 5 and X is 6, what is the value of X?
12. If the average of 18.25, 21.75, 19.5 and W is 20, what is the value of W?
13. If the average of 9.75, 10.25, 8.50 and F is 10, what is the value of F?
14. The average of 5 tests is 46 and that of first 4 is 45. The 5th test is?
15. The average of 7 tests is 90 and that of first 6 is 95. The 7th test is?
16. Tahir buys two books for Rs. 400 each, and four books for 250 each, what will be the average price of books?

17. Ahmad buys one book for Rs. 44, two books for 65 each, and three books for Rs. 80 each, what will be the average price of books?

18. Seemab buys two articles for Rs. 15 each and three articles for 35 each. Find the average price of the articles.

19. Maham buys two diaries for Rs. 75 each, two diaries for Rs. 100 and two diaries for 125 each, what will be the average price of diaries?

20. Hamid buys two wall clocks for Rs. 150 each, three wall clocks for Rs. 200 and one wall clock for 300 each. Find the average price of the wall clocks.

21. Abbas's average on 4 tests is 45. What does he need on his 5th test to raise his average 50?

22. Ali's average on 5 tests is 55. What does he need on his 6th test to raise his average 57?

23. The average of 6 tests is 8. What is the score needed in 7th test for an average of 9?

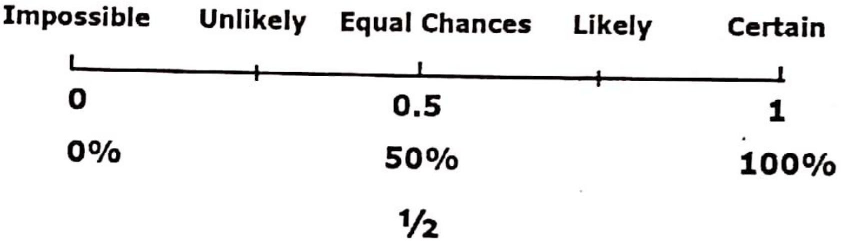
24. The average of 4 tests is 12. What is the score needed in 5th test for an average of 13?

25. Zuhra scored an average 34 runs in 3 matches. What does she need on her 4th test to raise her average 36?

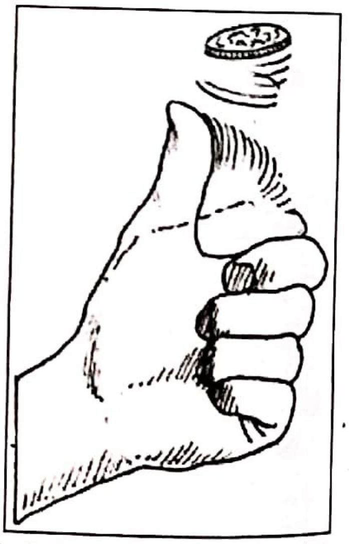
CHAPTER 26

PROBABILITY

Probability is the measure of the likelihood that an event will occur. Probability is quantified as a number between 0 and 1 (where 0 indicates impossibility and 1 indicates certainty). The higher the probability of an event, the more certain that the event will occur.



A simple example is the tossing of a fair (unbiased) coin. Since the coin is unbiased, the two outcomes ("head" and "tail") are both equally probable; the probability of "head" equals the probability of "tail." Since no other outcomes are possible, the probability is 1/2 (or 50%), of either "head" or "tail".



Probabilities are written as:

- Fractions from 0 to 1
- Percent from 0% to 100%
- Decimals from 0 to 1

Properties of Probability

- The probability of an event is always greater than or equal to 0
- The probability of an event lies between 0 and 1
- Probability of an impossible event is 0. That is $P(\Phi) = 0$.
- Probability of a sure event is 1. That is $P(S) = 1$.

An occurrence for which the outcome is uncertain is called an **experiment.** The set of all possible outcomes of an experiment is called **sample space** and is usually denoted with a capital S. The sample space for the coin-tossing experiment is **S = { H, T }**, where **H** symbolizes "heads up" and **T** stands for "tails up." **Events** are subsets of the sample space S. We denote an event by capital **E**. The subset **E = { H }** is the event of the coin landing heads up and **E = { T }** is the event of the coin landing tails up.

Computing Probability

Probability formula is the ratio of number of favourable outcomes to the total number of possible outcomes.

$$P(E) = \frac{\text{number of outcomes in event E}}{\text{total number of possible outcomes}} = \frac{n(E)}{n(S)}$$

Let's use the formula for the coin-tossing experiment.

The probability of the coin landing heads up is:

$$P(E) = \frac{\text{number of outcomes resulting in heads up}}{\text{total number of possible outcomes}} = \frac{1}{2}$$

The probability of the coin landing tails up is:

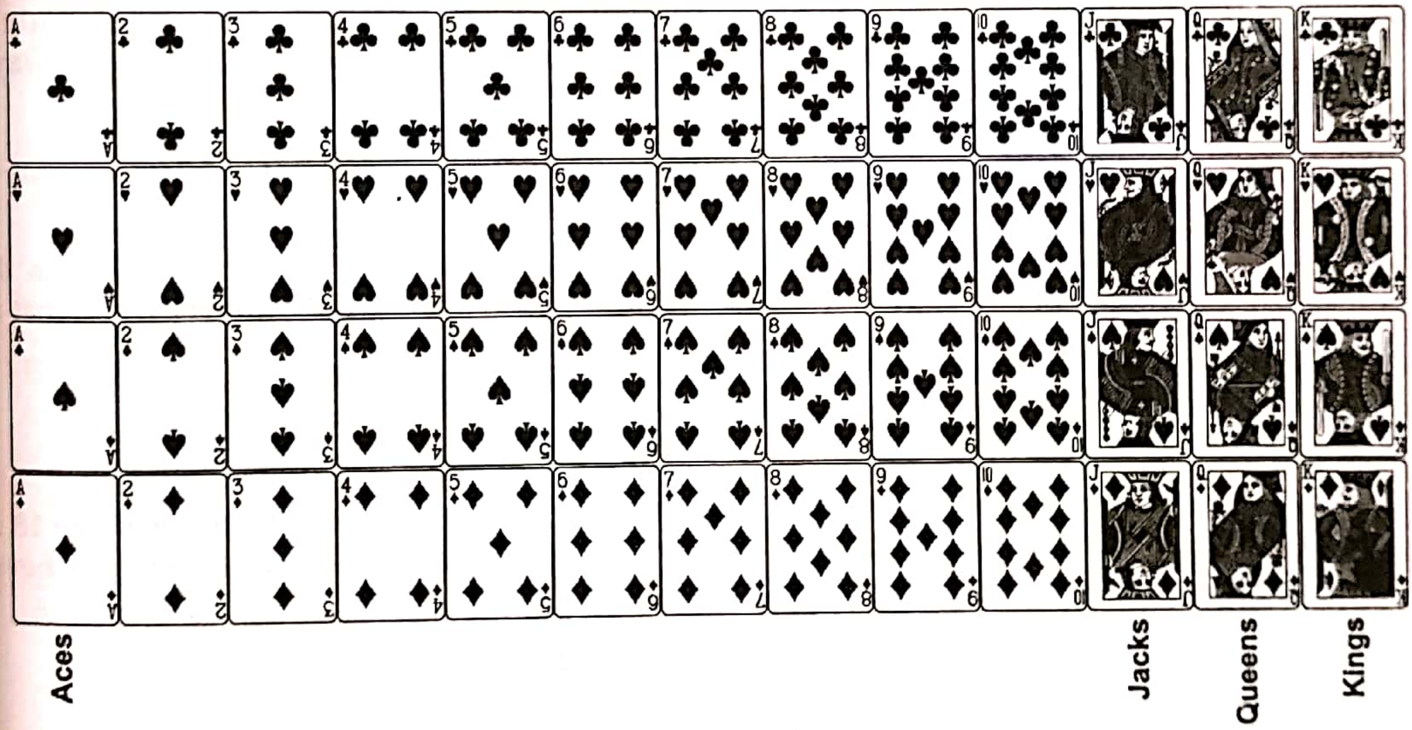
$$P(E) = \frac{\text{number of outcomes resulting in tails up}}{\text{total number of possible outcomes}} = \frac{1}{2}$$

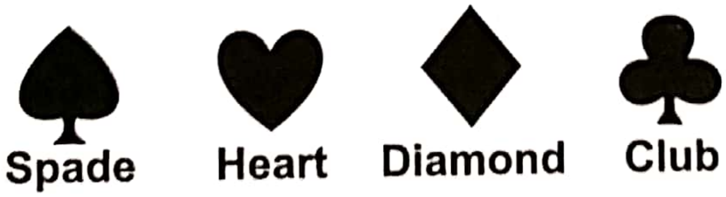
Note: The sum of the probabilities of all possible outcomes in the sample space equals 1.

Example 1: You are drawing one card from a standard 52-card deck. Find the probability of getting:

- **(a)** a queen
- **(b)** a diamond
- **(c)** a queen of diamonds
- **(d)** a black card
- **(e)** a black king

First of all, let's look at a picture of a standard 52-card deck.





Please note that a standard 52-card deck has 13 spades, 13 hearts, 13 diamonds, and 13 clubs!

Since we are told that the deck contains 52 cards, the number of outcomes in the sample space is $n(S)=52$, which will be used as the denominator of each probability fraction.

(a) Let's find the event of being dealt a queen. There are four queens in the deck. See picture above. Therefore, this event can occur in 4 different ways, that is, $n(E)=4$

$$P(queen)=\frac{4}{52}=\frac{1}{13}$$

(b) Let's find the event of being dealt a diamond. There are thirteen diamonds in the deck. See picture above. Therefore, this event can occur in 13 different ways, that is, $n(E)=13$

$$P(diamond)=\frac{13}{52}=\frac{1}{4}$$

(c) Let's find the event of being dealt a queen of diamonds. There is only one card in the deck that is a queen of diamonds. See picture above. Therefore, this event can occur in exactly 1 way, that is, $n(E)=1$.

$$P(queen\ of\ diamonds)=\frac{1}{52}$$

(d) Let's find the event of being dealt a black card. There are twenty-six black cards in the deck. See picture above. Therefore, this event can occur in 26 different ways, that is, $n(E)=26$.

$$P(black\ card)=\frac{26}{52}=\frac{1}{2}$$

(e) Let's find the event of being dealt a black king. There are two black kings in the deck. See picture above. Therefore, this event can occur in 2 different ways, that is, $n(E)=2$.

$$P(blackking)=\frac{2}{52}=\frac{1}{26}$$

Example 2: The U.S. Census Bureau claims that there are 69.2 million individuals over age 18 in the U.S. who have never been married. Find the empirical probability of selecting a never married person from the U.S. population of 235.8 million.

$$P(E)=\frac{69.2}{235.8}\approx 0.29$$

The empirical probability of selecting a never married person from the US population is approximately 0.29.

Example 3: Suppose we draw a card from a deck of playing cards. What is the probability that we draw a spade?

The sample space of this experiment consists of 52 cards, and the probability of each sample point is 1/52. Since there are 13 spades in the deck, the probability of drawing a spade is

P (Spade) = (13) (1/52) = 1/4

Example 4: Suppose a coin is flipped 3 times. What is the probability of getting two tails and one head?

For this experiment, the sample space consists of 8 sample points.

S = {TTT, TTH, THT, THH, HTT, HTH, HHT, HHH}

Each sample point is equally likely to occur, so the probability of getting any particular sample point is 1/8. The event "getting two tails and one head" consists of the following subset of the sample space.

A = {TTH, THT, HTT}

The probability of Event A is the sum of the probabilities of the sample points in A. Therefore,

P (a) = 1/8 + 1/8 + 1/8 = 3/8

PROBABILITY OF AN EVENT NOT OCCURRING

If we know P(E), the probability of an event E, then we can find the probability that the even will NOT occur, denoted P(not E). The event "not E" is called the **complement** of E.
The formula is

$$P(not\, E) = 1 - P(E)$$

This can be explained with the example of rain. If there is a probability of raining is 80% then probability of not raining is 20%. We can calculate this with the formula also

P (not Raining) = 1 – P (raining)
= 1 – 0.8
= 0.2 => 20%

Example 5: If you are dealt one card from a standard 52-card deck, find the probability that you are not dealt a king.

Since $P(not\, E) = 1 - P(E)$ then $P(not\, king) = 1 - P(king)$

There are four kings in a deck of 52 cards. The probability that you are dealt a king is $\frac{4}{52} = \frac{1}{13}$.

Therefore, the probability at you are not dealt a king is $1 - \frac{1}{13} = \frac{12}{13}$

Example 6: Let A be the event of rolling 4 on a die. What's the probability of not rolling 4?

The probability that we don't roll 4 is equal to:

$$1 - (\text{probability of rolling } 4) = 1 - \frac{1}{6} = \frac{5}{6}$$

"OR" OR UNIONS

"Or" Probability with Events that are Mutually Exclusive

Events that are **mutually exclusive** occur when it is impossible for events A and B to occur simultaneously. Another word that means mutually exclusive is **disjoint**. Examples:

- Turning left and turning right are Mutually Exclusive (you can't do both at the same time)
- Tossing a coin: Heads and Tails are Mutually Exclusive
- Cards: Kings and Aces are Mutually Exclusive or disjoint

If A and B are events that are mutually exclusive, then:

$$P(A \text{ or } B) = P(A) + P(B).$$

"Or" Probability with Events that are NOT Mutually Exclusive

Events that are **NOT mutually exclusive events (Inclusive)** occur when it is possible for events A and B to occur simultaneously. What is not Mutually Exclusive?

- Turning left and scratching your head can happen at the same time
- Kings and Hearts, because we can have a King of Hearts!

If A and B events that are **NOT** mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

Example 7: If you are selecting just one card from a standard 52-card deck, find the probability that you are selecting a jack or a ten.
There are four jacks and four tens in a deck of 52 cards.

The probability that you select a jack is $\frac{4}{52} = \frac{1}{13}$.

The probability that you select a ten is also $\frac{13}{52} = \frac{1}{13}$.

Therefore, the probability that you either select a jack or a ten is

$$P(\text{jack or ten}) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}.$$

Example 8: What is the probability of a die showing a 2 or a 5?

$$P(2) = \frac{1}{6}$$

$$P(5) = \frac{1}{6}$$

$$P(2 \text{ or } 5) = P(2) + P(5)$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

The probability of a die showing 2 or 5 is $\frac{1}{3}$

Example 9: If you are selecting just one card from a standard 52-card deck, find the probability that you are selecting a jack or a heart.

There are four jacks and thirteen hearts in a deck of 52 cards. There is also one jack of hearts in the deck.

The probability that you select a jack is $\frac{4}{52} = \frac{1}{13}$

The probability that you select a heart is $\frac{13}{52} = \frac{1}{4}$

The probability that you select the jack of hearts, P(A and B), is $\frac{1}{52}$.

Therefore, the probability at you either select a jack or a heart is

$$P(\text{jack or heart}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Example 10: The probabilities of three teams A, B and C winning a badminton competition are $\frac{1}{3}, \frac{1}{5}$ and $\frac{1}{9}$ respectively. Calculate the probability that

- **(a)** either A or B will win
- **(b)** either A or B or C will win
- **(c)** none of these teams will win
- **(d)** neither A nor B will win

(a) P (A or B will win) $= \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$

(b) P(Aor B or C will win) $= \frac{1}{3} + \frac{1}{5} + \frac{1}{9} = \frac{29}{45}$

(c) P(none will win) =1-P(A or B or C will win) $= 1 - \frac{29}{45} = \frac{28}{45}$

(d) P (neither A nor B will win) = 1-P(either A or B will win) = $= 1 - \frac{8}{15} = \frac{7}{15}$

"AND" OR INTERSECTIONS

"And" Probability with Independent Events

Two events, A and B, are independent if the fact that A occurs does not affect the probability of B occurring. Some examples of independent events are:

- Landing on heads after tossing a coin AND rolling a 5 on a single 6-sided die. Both tossing a coin and rolling a die are independent of each other and cannot affect the outcome.
- Choosing a 3 from a deck of cards, replacing it, AND then choosing an ace as the second card.

To find the probability of two independent events that occur in sequence, find the probability of each event occurring separately, and then multiply the probabilities. If A and B are events that are independent, then:

$$P\,(A \text{ and } B) = P\,(a)\,P\,(b)$$

Or

$$P\,(A \cap B) = P\,(a)\,P\,(b)$$

"And" Probability with Dependent Events

Events are said to be **dependent** if the occurrence of one of them has an effect on the probability of the other. If A and B are events that are dependent, then

$$P(A\, and\, B) = (A).P(B\, given\ that\ A\, has\ occurred).$$

Example 11: If you toss a quarter twice, find the probability that you get tails on the first and the second toss.

The equally likely outcome of tossing a quarter is either heads or tails. The probability for either one is 1/2.

Therefore, P (tails and tails) $= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Example 12: If you are selecting two cards from a standard 52-card deck, find the probability that you are selecting two jacks.

There are four jacks in a deck of 52 cards. The probability that you select a jack first is: 4/52 => 1/13.

Once you select a jack, there are 3 jacks left in a deck of 51 cards. The probability that you select a jack again is $\frac{3}{51} = \frac{1}{17}$.

Therefore, the probability at you select two jacks is:

$$P(jack\, and\ jack) = \frac{1}{13}.\frac{1}{17} = \frac{1}{221} \approx 0.00452$$

SAMPLING WITH REPLACEMENT AND WITHOUT REPLACEMENT

When we sample **with replacement**, the two sample values are independent. Practically, this means that what we get on the first one doesn't affect what we get on the second. Whenever we say probability with replacement, this implies that the events that are take in the probability are completely independent of each other.

Sampling is called **without replacement** when a unit is selected at random from the population and it is not returned to the main lot. First unit is selected out of a population of size N and the second unit is selected out of the remaining population of N−1 units and so on. In sampling without replacement, the two sample values aren't independent. Practically, this means that what we got for the first one affects what we can get for the second one.

Example 13: In a shipment of 20 computers, 3 are defective. Three computers are randomly selected and tested. What is the probability that all three are defective if the first and second ones are not replaced after being tested?

Probabilities: P(3 defectives) =

$$\frac{3}{20}\cdot\frac{2}{19}\cdot\frac{1}{18}=\frac{6}{6840}=\frac{1}{1140}$$

Example 14: Four cards are chosen at random from a deck of 52 cards without replacement. What is the probability of choosing a ten, a nine, an eight and a seven in order?

Probabilities: P(10 and 9 and 8 and 7) =

$$\frac{4}{52}\cdot\frac{4}{51}\cdot\frac{4}{50}\cdot\frac{4}{49}=\frac{256}{6,497,400}=\frac{32}{812,175}$$

Example 15: Three cards are chosen at random from a deck of 52 cards without replacement. What is the probability of choosing 3 aces?

Probabilities: P (3 aces) =

$$\frac{4}{52}\cdot\frac{3}{51}\cdot\frac{2}{50}=\frac{24}{132,600}=\frac{1}{5,525}$$

Conditional Probability

The conditional probability of an event B in relationship to an event A is the probability that event B occurs given that event A has already occurred. The notation for conditional probability is P(B | A), read as the probability of B given A.

Example 16: A math teacher gave her class two tests. 25% of the class passed both tests and 42% of the class passed the first test. What percent of those who passed the first test also passed the second test?

This problem describes a conditional probability since it asks us to find the probability that the second test was passed given that the first test was passed.

Formula for conditional probability is

$$P(B\mid A)=\frac{P(A\text{ and }B)}{P(A)}$$

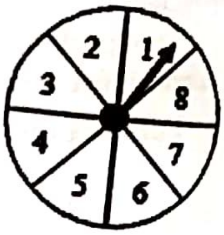
$$P(\text{Second}\mid\text{First})=\frac{P(\text{First and Second})}{P(\text{First})}=\frac{0.25}{0.42}=0.60=60\%$$

Example 17: A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

$$P(\text{White} \mid \text{Black}) = \frac{P(\text{Black and White})}{P(\text{Black})} = \frac{0.34}{0.47} = 0.72 = 72\%$$

SELF-EXERCISE

1. A die is rolled, find the probability that an even number is obtained.
2. Two coins are tossed, find the probability that two heads are obtained.
3. Which of these numbers cannot be a probability?
   - **(a)** -0.00001 **(b)** 0.5 **(c)** 1.001 **(d)** 0
   - **(e)** 1 **(f)** 20%
4. Two dice are rolled, find the probability that the sum is:
   - **(a)** equal to 1 **(b)** equal to 4 **(c)** less than 13
5. A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.
6. Four cards are chosen at random from a deck of 52 cards without replacement. What is the probability of choosing a ten, a nine, an eight and a seven in order?
7. One card is drawn from a standard pack of 52 playing cards. What is the probability of **(a)** picking a red card **(b)** picking a king **(c)** picking a diamond?
8. A jar contains 3 red marbles, 7 green marbles and 10 white marbles. If a marble is drawn from the jar at random, what is the probability that this marble is white?
9. The blood groups of 200 people are distributed as follows: 50 have type A blood, 65 have B blood type, 70 have O blood type and 15 have type AB blood. If a person from this group is selected at random, what is the probability that this person has O blood type?
10. A coin is tossed three times. What is the probability that it lands on heads exactly one time?
11. A spinner, numbered 1–8, is spun once. What is the probability of spinning
    - **(a)** an EVEN number? __________
    - **(b)** a multiple of 3? __________
    - **(c)** a PRIME number? __________
    - **(d)** 9? __________



Questions 12 – 17 refer to a bag that contains 6 blue marbles, 7 green marbles, and 3 red marbles. Find:

12. What is the probability of selecting a red marble?
13. What is the probability of selecting a red or a green marble?

14. If the probability of selecting a blue marble is 3/8, what is the probability of not selecting a blue marble?

15. What is the probability of selecting a green marble, replacing that marble, and then selecting a blue marble?

16. What is the probability of selecting a green marble, without replacement, and then selecting a blue marble?

17. What is the probability of selecting a blue marble and then selecting another blue marble without replacement?

18. A coin is tossed and a die with numbers 1-6 is rolled. What is P(heads and 3)?

19. Two cards are selected from a deck of cards numbered 1 – 10. Once a card is selected, it is not replaced. What is P(two even numbers)?

20. Which of the following in NOT an example of independent events?

(a) Rolling a die and spinning a spinner

(b) Tossing a coin two times

(c) Picking two cards from a deck with replacement of first card

(d) Selecting two marbles one at a time without replacement

21. A club has 25 members, 20 boys and 5 girls. Two members are selected at random to serve as president and vice president. What is the probability that both will be girls?

22. One marble is randomly drawn and then replaced from a jar containing two white marbles and one black marble. A second marble is drawn. What is the probability of drawing a white and then a black?

23. Maria rolls a pair of dice. What is the probability that she obtains a sum that is either a multiple of 3 OR a multiple of 4?

24. Events A and B are independent. The P(a) = 3/5, and P(not B) = 2/3. What is P (A and B)?

25. Twenty-six squares of paper lettered A to Z are placed in a can. One letter is randomly chosen from the can and turns out to be the letter G. A second letter is drawn without putting the letter G back into the can. Which is the probability that it will be the letter F?

CHAPTER 27

ALGEBRA

Algebra is the branch of mathematics that uses letters in place of some unknown numbers.

The letters used in algebra can either stand for **variables** (the value of the letter can change, like in the examples of the area of a rectangle and the area of a square) or **constants** (where the value does not change), for example pi.

Terms and Factors

A **term** in an algebraic expression is an expression involving letters and/or numbers. (called **factors**), multiplied together. The algebraic expression 5x is an example of one single **term**. It has **factors** 5 and x. The 5 is called the **coefficient** of the term and the x is a **variable.** 5x + 3y has two **terms**.

"Like terms" are terms that contain the **same variables** raised to the **same power.**

$3x^2$ and $7x^2$ are **like terms.**

$-8x^2$ and $5y^2$ are **not like terms,** because the variable is not the same.

Algebraic Expression or Polynomial

An algebraic expression is a mathematical phrase that can contain ordinary numbers, variables (like x or y) and operators (like add, subtract, multiply, and divide). Here are some algebraic expressions:

$a + 1$

$a – b$

$3x$

$x - a / b$

In the above expression the "/" means divide. The "3x" means multiply the variable x by 3. Many English phrases translate directly into algebraic expressions, as shown in the following table (where x stands for "a number"):

English expression	Algebraic expression
A number plus eight	$x + 8$
four less than a number	$x - 4$
half of a number	$x / 2$

If you set an algebraic expression equal to something, with an equal sign, you now have an equation, like: $X - 4 = 10$.

An Algebraic expression is also called Polynomial. There are special names for polynomials with one, two or three terms that is monomial, binomial and trinomial respectively. Example: are $3xy^2, 5x-1$ and $3x+5y^2-3$ are monomial, binomial and trinomial respectively.

SIMPLIFYING ALGEBRAIC EXPRESSIONS

By "simplifying" an algebraic expression, we mean writing it in the most compact or efficient manner, without changing the value of the expression. This mainly involves *collecting like terms*, which means that we add together anything that can be added together. The rule here is that only *like* terms can be added together.

Adding and Subtracting Terms

Combining like terms is permitted because of the distributive law. For example,

$$3x^2 + 5x^2 = (3+5)x^2 = 8x^2$$

Example 1: $x^2 + 2x + 3x^2 + 2 + 4x + 7$

Starting with the highest power of x, we see that there are four x-squareds in all $(1x^2 + 3x^2)$. Then, we collect the first powers of x, and see that there are six of them $(2x + 4x)$. The only thing left is the constants $2 + 7 = 9$. Putting this all together we get

$$x^2 + 2x + 3x^2 + 2 + 4x + 7$$
$$= 4x^2 + 6x + 9$$

Example 2 : $3x + 2(x-4)$

$$= 3x + 2x - 8$$
$$= 5x - 8$$

Negative signs in front of parentheses

A special case is when a minus sign appears in front of parentheses. At first glance, it looks as though there is no factor multiplying the parentheses, and you may be tempted to just remove the parentheses. What you need to remember is that the minus sign indicating subtraction should always be thought of as adding the opposite. This means that you want to add the opposite of the entire thing inside the parentheses, and so you have to change the sign of each term in the parentheses.

$$3x - (2 - x)$$
$$= 3x - 2 + x$$
$$= 4x - 2$$

However, if there is only a plus sign in front of the parentheses, then you can simply erase the parentheses:

$$3x + (2 - x)$$
$$= 3x + 2 - x$$

Example 3: Simplify $-5\,[-2\,(m - 3n) + 4n]$

Go back to Order of Operations if you are not sure what to do first with this question.

$$-5[-2(m - 3n) + 4n]$$

The first thing we do is expand out the round brackets inside.

$$-2(m - 3n) = -2m + 6n$$

The negative times negative in the middle gives positive 6n.

Now add the 4n in the square brackets:

$$[-2m + 6n + 4n] = [-2m + 10n]$$

Remembering the –5 out front, our problem has become:

$$-5\,[-2m + 10n] = 10m - 50n$$

Example 4: Simplify $-[7(a - 2b) - 4b]$

$$\begin{aligned} &-[7(a - 2b) - 4b] \\ &= -[7a - 14b - 4b] \\ &= -[7a - 18b] \\ &= -7a + 18b \end{aligned}$$

Multiplication of Algebraic Expressions

When we multiply algebraic expressions, we need to remember the Exponent Rules.

Example of an Exponent

For the example 5^3, we say that:

5 is the **base** and 3 is the **index** (or **power**, or **exponent**).

5^3 means "multiply 5 by itself 3 times".

Let's see how algebra multiplication works with a series of examples.

Example 1: Multiply $x^3(x^4 + 5a)$

$$\begin{aligned} &x^3(x^4 + 5a) \\ &= x^7 + 5ax^3 \end{aligned}$$

We cannot simplify this answer any further. We present our answer in alphabetical order since it makes it easier to read.

Example 2: Multiply $(x + 5)(a - 6)$

We multiply this out as follows. We take each term of the first bracket and multiply them by the second bracket. Then we expand out the result.

$$\begin{aligned} &(x + 5)(a - 6) \\ &= x(a - 6) + 5(a - 6) \\ &= ax - 6x + 5a - 30 \end{aligned}$$

We cannot do any more with this answer. There are no like terms, so we cannot simplify it in any way.

Example 3: Multiply $(2x + 3)(x^2 - x - 5)$

We take the 2 terms of the first bracket and multiply both of them by the second bracket.

$$\begin{aligned} &(2x + 3)(x^2 - x - 5) \\ &= (2x)(x^2 - x - 5) + (3)(x^2 - x - 5) \\ &= (2x^3 - 2x^2 - 10x) + (3x^2 - 3x - 15) \end{aligned}$$

This time we could collect together some like terms. There is:

$$-2x^2 + 3x^2 = x^2 \text{ And } -10x - 3x = -13x$$
$$= 2x^3 + x^2 - 13x - 15$$

Example 4: Simplify $(x-3)^2$

$$(x-3)^2$$
$$= (x-3)(x-3)$$
$$= x(x-3) - 3(x-3)$$
$$= x^2 - 3x - 3x + 9$$
$$= x^2 - 6x + 9$$

Example 5: Simplify $5x[-4 + 10(x - y)] + 7x$

$$5x[-4 + 10(x-y)] + 7x$$
$$= 5x[-4 + 10x - 10y] + 7x$$
$$= -20x + 50x^2 - 50xy + 7x$$
$$= -13x + 50x^2 - 50xy$$

Division of Algebraic Expressions

Our first examples of division of algebraic expressions involve simplifying and cancelling.

Example 1: Simplify $\frac{3ab(4a^2b^5)}{8a^2b^3}$

First, we multiply out the top line: $\frac{12a^3b^6}{8a^2b^3}$

When we write it out in full, this means: $\frac{12 \times aaa \times bbbbbb}{8 \times aa \times bbb}$

Next, cancel the numbers top and bottom (we divide top and bottom by 4), the "a" terms (we cancel $a^2 = aa$ from top and bottom) and the "b" terms (we cancel b^3 = bbb from top and bottom) to give us the final answer:

$$\frac{3ab^3}{2}$$

Example 2: Simplify $\frac{12m^2n^3}{(6m^4n^5)^2}$

We square the denominator (bottom) of the fraction:

$$\frac{12m^2n^3}{(6m^4n^5)^2} = \frac{12m^2n^3}{36m^8n^{10}}$$

Next, we cancel out the numbers, and the "m" and "n" terms to give the final answer:

$$\frac{1}{3m^6n^7}$$

Example 3: Simplify $\dfrac{6p^3q^2-10p^2q}{4q}$

With this example, we'll break it into 2 fractions, both with denominator 4q to make it easier to see what to do.

$$\frac{6p^3q^2-10p^2q}{4q}=\frac{6p^3q^2}{4q}-\frac{10p^2q}{4q}$$

Next, we cancel the numbers and variables: $\dfrac{3p^3q}{2}-\dfrac{5p^2}{2}$

Finally, we combine the fractions: $\dfrac{3p^3q-5p^2}{2}$

After you have had some practice with these, you'll be able to do it without separating them into 2 fractions first.

Example 4: Simplify $\dfrac{3+\dfrac{1}{x}}{\dfrac{5}{x}+4}$

Method 1- Multiplying by the Reciprocal

I take the top expression (numerator) and turn it into a single fraction with denominator x.

$$3+\frac{1}{x}=\frac{3x+1}{x}$$

We do likewise with the bottom expression (denominator).

$$\frac{5}{x}+4=\frac{5+4x}{x}$$

So the question has become:

$$\frac{3+\dfrac{1}{x}}{\dfrac{5}{x}+4}=\frac{\dfrac{3x+1}{x}}{\dfrac{5+4x}{x}}$$

We think of the right side as a division of the top by the bottom: $\dfrac{3x+1}{x}\div\dfrac{5+4x}{x}$

To divide by a fraction, you multiply by the reciprocal: $\dfrac{3x+1}{x}\times\dfrac{x}{5+4x}=\dfrac{3x+1}{5+4x}$

The x's cancelled out, and we have our final answer, which cannot be simplified any more.

Method 2 - Multiplying Top and Bottom

I recognise that I have "/x" in both the numerator and denominator. So if I just multiply top and bottom by x, it will simplify everything by removing the fractions on top and bottom.

$$\frac{3+\frac{1}{x}}{\frac{5}{x}+4}\times\frac{x}{x}$$

I am really just multiplying by "1" and not changing the original value of the fraction — just changing its form.

So I multiply each element of the top by x and each element of the bottom by x and I get:

$$\frac{3+\frac{1}{x}}{\frac{5}{x}+4}\times\frac{x}{x}=\frac{3x+1}{5+4x}$$

Our result means that the answer is 1813 with remainder 7, or:

$$\frac{23576}{13}=1813\frac{7}{13}$$

We use a similar technique for long division in algebra.

Example 6 - Algebraic Long Division

Simplify $(3x^2-11x-4)\div(x-4)$

We are dividing a polynomial of degree 2 by a polynomial of degree 1. This is algebraic long division.

Step 1: We look at the first term of $(3x^2 - 11x - 4)$ and the first term of $(x - 4)$.

Divide as follows: $3x^2 \div x = 3x$

We write $3x$ at top of our long division and multiply $(3x)(x - 4) = 3x^2 - 12x$ to give the second row of our solution.

$$\begin{array}{r|l} & \quad 3x \\ \hline x-4 & 3x^2-11x-4 \\ & \underline{3x^2-12x} \end{array}$$

Step 2: Subtracting the second row from the first gives:

$$\begin{array}{r|l} & \quad 3x \\ \hline x-4 & 3x^2-11x-4 \\ & \underline{3x^2-12x} \\ & \qquad\quad x \end{array}$$

Be careful with

$$-11x - (-12x) = -11x + 12x = x$$

Step 3: Bring down the -4 from the first row:

$$\begin{array}{rr} & 3x \\ x-4 & \overline{)\,3x^2-11x-4} \\ & \underline{3x^2-12x}\ \downarrow \\ & x\ -\ 4 \end{array}$$

Step 4: Next, multiply (1) by $(x-4)$ to get the 4th row.

$$\begin{array}{rr} & 3x\ +\ 1 \\ x-4 & \overline{)\,3x^2-11x-4} \\ & \underline{3x^2-12x} \\ & x\ -\ 4 \\ & \underline{x\ -\ 4} \end{array}$$

Step 5: Subtract the 4th row from the 3rd:

$$\begin{array}{rr} & 3x\ +\ 1 \\ x-4 & \overline{)\,3x^2-11x-4} \\ & \underline{3x^2-12x} \\ & x\ -\ 4 \\ & \underline{x\ -\ 4} \\ & 0 \end{array}$$

So $(3x^2 - 11x - 4) \div (x - 4) = 3x + 1$

You can check your answer by multiplying $(3x + 1)$ by $(x - 4)$ and you'll get $(3x^2 - 11x - 4)$.

Example 6: Simplify $\frac{6x^2+6+7x}{2x+1}$

We can rewrite $\frac{6x^2+6+7x}{2x+1}$ as $(6x^2 + 7x + 6) \div (2x + 1)$

Once again we are dividing a polynomial of degree 2 by a polynomial of lower degree (1). This is algebraic long division.

Step 1: $6x^2 \div 2x = 3x$

So we write the following, using $(3x)(2x + 1) = 6x^2 + 3x$ for the second row:

$$\begin{array}{rl} & 3x \\ 2x+1 & \overline{)\,6x^2+7x+6} \\ & \underline{6x^2+3x} \end{array}$$

Step 2: We subtract $6x^2 + 3x$ from the first row:

$$\begin{array}{r|l}
 & 3x \\ \hline
2x+1 & 6x^2+7x+6 \\
 & \underline{6x^2+3x} \\
 & 4x
\end{array}$$

Step 3: Bring down the 6:

$$\begin{array}{r|l}
 & 3x \\ \hline
2x+1 & 6x^2+7x+6 \\
 & \underline{6x^2+3x} \quad \downarrow \\
 & 4x+6
\end{array}$$

Step 4: Divide $4x$ by $2x$. Our answer is 2 and we multiply $2(2x + 1)$ to get the 4th row.

$$\begin{array}{r|l}
 & 3x+2 \\ \hline
2x+1 & 6x^2+7x+6 \\
 & \underline{6x^2+3x} \\
 & 4x+6 \\
 & \underline{4x+2}
\end{array}$$

Step 5: Subtract, and we are left with 4.

$$\begin{array}{r|l}
 & 3x+2 \\ \hline
2x+1 & 6x^2+7x+6 \\
 & \underline{6x^2+3x} \\
 & 4x+6 \\
 & \underline{4x+2} \\
 & 4
\end{array}$$

So the answer is: $3x+2\left(\frac{4}{2x+1}\right)$

REMAINDERS

Remainder is the amount left over after division when one divisor does not divide the dividend exactly

Example 1: Find the quotient and the remainder of the division of $x^2 - 2x^2 - 4$ the *dividend*, by $x - 3$ the *divisor*.

The dividend is first rewritten like this: x^3-2x^2+0x-4

The quotient and remainder can then be determined as follows:

Divide the first term of the dividend by the highest term of the divisor (meaning the one with the highest power of x, which in this case is x). Place the result above the bar ($x^3 \div x = x^2$).

$$\begin{array}{rl} & x^2 \\ x-3 & \overline{\smash{)}\,x^3-2x^2+0x-4} \end{array}$$

Multiply the divisor by the result just obtained (the first term of the eventual quotient). Write the result under the first two terms of the dividend ($x^2 \cdot (x-3) = x^3-3x^2$).

$$\begin{array}{rl} & x^2 \\ x-3 & \overline{\smash{)}\,x^3-2x^2+0x-4} \\ & \,x^3-3x^2 \end{array}$$

Subtract the product just obtained from the appropriate terms of the original dividend (being careful that subtracting something having a minus sign is equivalent to adding something having a plus sign), and write the result underneath ($(x^3-2x^2)-(x^3-3x^2) = -2x^2+3x^2 = x^2$) Then, "bring down" the next term from the dividend.

$$\begin{array}{rl} & x^2 \\ x-3 & \overline{\smash{)}\,x^3-2x^2+0x-4} \\ & \underline{\pm x^3 \mp 3x^2} \\ & \qquad +x^2+0x \end{array}$$

Repeat the previous three steps, except this time use the two terms that have just been written as the dividend.

$$\begin{array}{rl} & x^2+x \\ x-3 & \overline{\smash{)}\,x^3-2x^2+0x-4} \\ & \underline{\pm x^3 \mp 3x^2} \\ & \qquad +x^2+0x \\ & \qquad \underline{\pm x^2 \mp 3x} \\ & \qquad\qquad\quad +3x-4 \end{array}$$

Repeat step 4. This time, there is nothing to "pull down".

$$\begin{array}{r l}
 & x^2 + x \ \ + 3 \\ \cline{2-2}
x-3\big) & x^3 - 2x^2 + 0x - 4 \\
 & \underline{\pm x^3 \mp 3x^2} \\
 & \qquad + x^2 + 0x \\
 & \qquad \underline{\pm x^2 \mp 3x} \\
 & \qquad\qquad\quad + 3x - 4 \\
 & \qquad\qquad\quad \underline{\pm 3x \mp 9} \\
 & \qquad\qquad\qquad\quad + 5
\end{array}$$

The polynomial above the bar is the quotient $q(x)$, and the number left over (5) is the remainder $r(x)$.

Example 2: $p(x) = x^3 - 7x - 6$, and let's divide by the linear factor $x - 4$ (so $a = 4$):

$$\begin{array}{r l}
 & x^2 + 4x \ \ + 9 \\ \cline{2-2}
x-4\big) & \ \ x^3 + 0x^2 - 7x - 6 \\
 & \underline{\pm x^3 \mp 4x^2} \\
 & \qquad 4x^2 - 7x - 6 \\
 & \quad \underline{\pm 4x^2 \mp 16x \quad} \\
 & \qquad\qquad\qquad 9x - 6 \\
 & \qquad\qquad\quad\ \ \underline{\pm 9x \mp 36} \\
 & \qquad\qquad\qquad\qquad 30
\end{array}$$

So we get a quotient of $q(x) = x^2 + 4x + 9$ on top, with a remainder of $r(x) = 30$.

EQUATIONS

Algebraic equation

An algebraic equation is an equation that includes one or more variables.

Example 1. Hadya has \$17 in her bank. How much money does she need to buy a game that costs \$68?

Let x represent the amount of money Hadya needs. Then the following equation can represent this problem:

$$17 + x = 68$$

We can subtract 17 from both sides of the equation to find the value of x.

$$68 - 17 = x$$

$x = 51$, so Hadya needs \$51 to buy the game.

In the problem above, x is a **variable**. The symbols $17 + x = 68$ form an **algebraic equation**. Let's look at some examples of writing algebraic equations.

Example 2: Write each sentence as an algebraic equation.

Sentence	Algebraic Equation
A number increased by nine is fifteen.	$y + 9 = 15$
Twice a number is eighteen.	$2n = 18$
Four less than a number is twenty.	$x - 4 = 20$
A number divided by six is eight.	$K/6 = 8$

Example 3: Write each sentence as an algebraic equation.

Sentence	Algebraic Equation
Twice a number, decreased by twenty-nine, is seven.	$2t - 29 = 7$
Thirty-two is twice a number increased by eight.	$32 = 2a + 8$
The quotient of fifty and five more than a number is ten.	$\frac{50}{n+5} = 10$
Twelve is sixteen less than four times a number.	$12 = 4x - 16$

Example 4: Write each sentence as an algebraic equation.

Sentence	Algebraic Equation
Rabia is x years old. In thirteen years she will be twenty-four years old.	$x + 13 = 24$
Each piece of candy costs 25 cents. The price of h pieces of candy is $2.00.	$25h = 200$ or $.25h = 2.00$
Fatima made a withdrawal of d dollars from her savings account. Her old balance was $350, and her new balance is $280.	$350 - d = 280$
A large pizza pie with 15 slices is shared among p students so that each student's share is 3 slices.	$\frac{15}{p} = 3$

Types of Algebraic equations

Type of Equation	General Form	Description	Example
Linear	$y = ax + b$	Equation with variables but no exponents	$y = x - 1$
Quadratic	$y = ax^2 + bx + c$	Equation with a squared term	$y = 2x^2 - x + 5$
Radical	$y = \sqrt{x}$	Equation with exponent 1212	$y = \sqrt{50}$
Rational	$y = \frac{a}{x} + b$	Equation with a variable in the denominator	$y = \frac{2}{x+1}$
Exponential	$y = a^x$	Equation with a variable as an exponent	$y = 2^x$

Solving Equations

$$\boxed{?}+5=7$$

We just need to figure out which number should go into the box to make it a true statement. Clearly, we need to replace the question mark with "2":

$$\boxed{2}+5=7$$

Solving equations using algebra is really no different. Instead of using a **box,** we use a **letter** to represent a number. Our task is to find the correct number (or sometimes there may be more than one number) that makes the equation true. The process is:

1. We are aiming to get x (or whatever letter the question uses).
2. We solve equations by **balancing:** whatever we do to one side of an equation, we must do the same to the other side. So if we add 4 to the left hand side, we must add 4 to the right hand side as well. If we multiply on the left side by 2, we multiply on the right side by 2 as well.

Example 5: Solve the equation $x-6=10$

We need to "get rid of" the -6 on the left hand side so we are left with x only on the left hand side.

The opposite of subtracting 6 is adding 6.

If we add 6 to both sides, we will remove the -6 on the left.

$$x-6=10$$
$$x-6+6=10+6$$
$$x=16$$

So, the value of x needs to be 16 to make the equation true.

CHECK into the original question:

$16-6=10$. It checks out okay.

Example 6: Solve $\frac{3x}{4}=7$

This time we need to do 2 steps to solve the equation. We notice there is a 4 on the bottom of the fraction.

$$\frac{3x}{4}=7$$

This is equivalent to dividing by 4. The opposite to dividing by 4 is multiplying by 4. So, we do that first:

$$\frac{3x}{4}\times 4=7\times 4$$

Cancelling the 4s on the left gives: $3x=28$

Now we need to divide both sides by 3, since we have a "3x" on the left hand side of the equation.

$$x = \frac{28}{3} = 9\frac{1}{3}$$

CHECK is our answer correct?

Substituting our answer in the left hand side gives:

$$\frac{3x}{4} = \frac{3}{4}x = \frac{3}{4} \times \frac{28}{3}$$

Cancelling the 3s (which gives us 1) and the 28 with the 4 gives us 7:

$$\frac{3}{4} \times \frac{28}{3} = 7$$

The right hand side in the question was 7, so we are confident our answer is correct.

Example 7: Solve $5 - (x + 2) = 5x$

First, we expand out the bracket.

$$5 - (x+2) = 5x$$
$$5 - x - 2 = 5x$$
$$3 - x = 5x$$

Now we recognize that it is easier to get all the x on the right side, by adding x to both sides:

$$3 = 6x$$

Now I divide both sides by 6 and swap the sides:

$$x = 0.5$$

We check our answer in both sides of the equation. If it works, it must be the right answer.

$$\text{LHS} = 5 - (0.5+2) = 2.5$$
$$\text{RHS} = 5 \times 0.5 = 2.5 = \text{LHS}.$$

Example 8: Solve $\frac{6x-7}{4} + \frac{3x-5}{7} = \frac{5x+78}{28}$

Get rid of the denominators by multiplying both sides by 28, the smallest number that 4, 7, and 28 will divide into evenly.

$$28\left(\frac{6x-7}{4} + \frac{3x-5}{7}\right) = 28\left(\frac{5x+78}{28}\right)$$

Simplify

$$28\left(\frac{6x-7}{4}\right) + 28\left(\frac{3x-5}{7}\right) = 28\left(\frac{5x+78}{28}\right)$$
$$7(6x-7) + 4(3x-5) = 1(5x+78)$$
$$42x - 49 + 12x - 20 = 5x + 78$$

Add the x terms and the constants on the left side of the equation.

$$54x-69=5x+78$$

Subtract $5x$ from both sides of the equation.

$$49x-69=78$$

Add 69 to both sides of the equation.

$$49x=147$$

Divide both sides by 49.

$$x=3$$

The answer is $x = 3$.

ALGEBRA WORD PROBLEMS

Example of algebra word problems are numerous. The goal of this section is to give you the skills that you need to solve a variety of these algebra word problems in CSS paper.

Example 1: A football team lost 5 yards and then gained 9. What is the team's progress?

For lost, use negative. For gain, use positive.

Progress = -5 + 9 = 4 yards

Example 2: Maria bought 10 notebooks and 5 pens costing 2 dollars each. How much did Maria pay?

$2 \times (10 + 5) = 2 \times 10 + 2 \times 5 = 20 + 10 = 30$ dollars

Example 3: A customer pays 50 dollars for a coffee maker after a discount of 20 dollars. What is the original price of the coffee maker?

Let x be the original price.

$$x - 20 = 50$$
$$x - 20 + 20 = 50 + 20$$
$$x + 0 = 70$$
$$x = 70$$

Example 4: Half a number plus 5 is 11.What is the number?

Let x be the number. Always replace "is" with an equal sign

$$(1/2)x + 5 = 11$$
$$(1/2)x + 5 - 5 = 11 - 5$$
$$(1/2)x = 6$$
$$2 \times (1/2)x = 6 \times 2$$
$$x = 12$$

Example 5: The sum of two consecutive even integers is 26. What are the two numbers?

Let 2n be the first even integer and let 2n + 2 be the second integer.

$$2n + 2n + 2 = 26$$
$$4n + 2 = 26$$

$4n + 2 - 2 = 26 - 2$

$4n = 24$

$n = 6$

So the first even integer is $2n = 2 \times 6 = 12$ and the second is $12 + 2 = 14$.

Below are more complicated algebra word problems.

Example 6: The ratio of two numbers is 5 to 1. The sum is 18. What are the two numbers?

Let x be the first number and y be the second number

$x / y = 5 / 1$

$x + y = 18$

Using $x / y = 5 / 1$, we get $x = 5y$ after doing cross multiplication

Replacing $x = 5y$ into $x + y = 18$, we get $5y + y = 18$

$6y = 18$

$y = 3$

$x = 5y = 5 \times 3 = 15$

As you can see, $15/3 = 5$, so ratio is correct and $3 + 15 = 18$, so the sum is correct.

Example 7: The area of a rectangle is $x^2 + 4x - 12$. What are the dimensions of the rectangle (length and width)?

The main idea is to factor $x^2 + 4x - 12$

Since $-12 = -2 \times 6$ and $-2 + 6 = 4$

$x^2 + 4x - 12 = (x + -2) \times (x + 6)$

Since the length is usually longer, length $= x + 6$ and width $= x + -2$

Example 8: The area of a rectangle is 24 cm². The width is two less than the length. What is the length and width of the rectangle?

Let x be the length and let $x - 2$ be the width.

Area = length × width $= x \times (x - 2) = 24$

$x \times (x - 2) = 24$

$x^2 + -2x = 24$

$x^2 + -2x - 24 = 0$

Since $-24 = 4 \times -6$ and $4 + -6 = -2$, we get:

$(x + 4)\ (x - 6) = 0$

This leads to two equations to solve:

$x + 4 = 0$ and $x - 6 = 0$

$x + 4 = 0$ gives $x = -4$

Reject this value since a dimension cannot be negative

$x - 6 = 0$ gives $x = 6$

Therefore, length $= 6$ and width $= x - 2 = 6 - 2 = 4$

Example 9: The sum of two numbers is 16. The difference is 4. What are the two numbers?

Let x be the first number and y be the second number

$$x + y = 16$$

$$x - y = 4$$

Solve the equation by elimination Method

Adding the left sides and the right sides gives:

$$x + x + y + -y = 16 + 4$$

$$2x = 20$$

$$x = 10$$

Since $x + y = 16$, $10 + y = 16$

$$10 + y = 16$$

$$10 - 10 + y = 16 - 10$$

$$y = 6$$

The numbers are 10 and 6.

Example 10. : The sum of two consecutive odd numbers is 52. What are the two odd numbers?

An odd number is 1 more (or 1 less) than an even number. And so we represent an odd number as 2n + 1.

Let 2n + 1, then, be the first odd number. Then, the next one will be 2 more -- it will be 2n + 3. The problem states that their sum is 52:

$$2n + 1 \ + \ 2n + 3 \ = 52$$

We will now solve that equation for n, and then replace the solution in 2n + 1 to find the first odd number. We have:

$$4n + 4 = 52$$

$$4n = 48$$

$$n = 12.$$

Therefore, the first odd number is $2(12) + 1 = 25$.

And so the next one is 27. Their sum is 52.

MULTIPLE CHOICE QUESTIONS WITH EXPLANATORY ANSWERS

In each of the following questions a number of possible answers are given, out of which one answer is correct. Find out the correct answer.

1. If $4x + 13 = 7 - 2x$, what is the value of x?

(a) $-\frac{10}{3}$ **(b)** -3 **(c)** -1 **(d)** 1

Ans. **(c)**

Solution: Add 2x to each side: $6x + 13 = 7$. Subtract 13 from each side: $6x = -6$. Divide each side by 6: $x = -1$.

2. If $\frac{x+3}{2} + 3x = 5(x-3) + \frac{x+23}{5}$, what is the value of x?

(a) 3 (b) 5 (c) 7 (d) 9

Ans. (c)

Solution:

Multiply both sides of the equation by 10 to get rid of fractions:

$$5(x+3)+30x=50(x-3)+2(x+23)$$

Use the distributive law to get rid of parentheses:

$$5x+15+30x=50x-150+2x+46$$

Combine like terms:

$$35x+15=52x-104$$

Get the variable onto one side:

$$15=17x-104$$

Add 104 to each side:

$$119=17x$$

Divide by 17:

$$x=7$$

3. **Which of the following is a solution of the equation $\sqrt{2x}+5=3$?**

(a) 2 (b) 4 (c) 16 (d) This equation has no solution.

Ans. (d)

Solution: $\sqrt{2x}+5=3 \Rightarrow \sqrt{2x}=-2$. Since by definition a square root cannot be negative, this equation has no solution.

4. **If $ax-b=c-dx$, what is the value of x in terms of a, b, c, and d?**

(a) $\frac{b+c}{a+d}$ (b) $\frac{c-b}{a-d}$ (c) $\frac{b+c-d}{a}$ (d) $\frac{c-b}{a+d}$

Ans. (a)

Solution: Treat a, b, c, and d as constants, and use the six-step method to solve for x:

$$ax-b=c-dx \Rightarrow ax-b+dx=c$$

$$\Rightarrow \quad ax+dx=c+b \Rightarrow x(a+d)=b+c$$

$$\Rightarrow \quad x=\frac{b+c}{a+d}$$

5. **If the sum of two numbers is 13 and the difference of the numbers is 23, what is the product of the numbers?**

(a) –90 (b) –23 (c) 0 (d) 18

Ans. (a)

Solution: Represent the numbers by x and y. Then $x+y=13$ and $x-y=23$. Now add the equations:

$$\begin{array}{l} x+y=13 \\ x-y=23 \\ \hline 2x \quad =36 \Rightarrow x=18 \end{array}$$

So $18 + y = 13 \Rightarrow y = -5$. Finally, the product $xy = (18)(-5) = -90$

6. **If $4^a = 8^b$, what is the ratio of a to b?**

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $\frac{2}{1}$

Ans. (c)

Solution: $4^a = (2^2)^a = 2^{2a}$ and $8^b = (2^3)^b = 2^{3b}$

Hence bases are equal therefore, $\Rightarrow 2a = 3b \Rightarrow \frac{a}{b} = \frac{3}{2}$

7. **Given $f(x) = x^2 + 2x - 1$, find $f(2)$.**

(a) 7 (b) 10 (c) 27 (d) 3

Ans (a)

Solution: $f(2) = (2)^2 + 2(2) - 1$

$= 4 + 4 - 1$

$= 7$

8. **Solve: $7x + 11y = 1, 8x + 13y = 2$.**

(a) $x = -5, y = 3$ (b) $x = -7, y = 5$ (c) $x = -3, y = 2$ (d) None of these

Ans. (c)

Solution: $7x + 11y = 1$ ...(1)

$8x + 13y = 2$ ...(2)

Multiplying Eq. (1) and Eq. (2) by 8 and 7, respectively, we have

$56x + 88y = 8$ and $56x + 91y = 14$

Subtracting the second equation from the first, we get $-3y = -6$, i.e. $y = 2$

From Eq. (1), $x = \frac{1-11(y)}{7} = \frac{1-11(2)}{7} = -3$

$x = -3$ and $y = 2$

9. **If $\frac{2}{x} + \frac{3}{y} = \frac{9}{xy}$ and $\frac{4}{x} + \frac{9}{y} = \frac{21}{xy}$, where $x \neq 0, y \neq 0$, the values of x and y are, respectively**

(a) 0 and 1 (b) 1 and 2 (c) 2 and 3 (d) 1 and 3

Ans. (d)

Solution: Multiplying each equation throughout by xy, we get

$2y + 3x = 9$ and, $4y + 9x = 21$

On solving these equations, we get

$x = 1, y = 3$

10. **Expansion of $(x + 3y)(x - y)$ gives**

(a) $x^2 + 2xy - 3y^2$ (b) $x + 3y - 2xy$ (c) $-2y + 3x$ (d) $3y - 42x$

Ans. (a)

INEQUALITIES

An **inequality** is a statement which involves one of the sign $<$, $>$, $\leq$ and $\geq$.

A number is positive if it is greater than 0, so $1, \frac{1}{1000}$, and 53.4 are all positive numbers. If you think of numbers as points on a number line, positive numbers correspond to points to the right of 0. A number is negative if it is less than 0 $-\frac{4}{5}, -50$, and $-.0001$ are all negative numbers. Negative numbers correspond to point to the left of 0 on a number line.

Inequalities have certain properties that are similar to equations. We can talk about the left side and the right side of inequality, and we can use algebraic expressions for the sides of an inequality. For example, $6x < 5x + 4$.

A value for an unknown satisfies an inequality, when you evaluate each side of the inequality the numbers satisfy the inequality. So if $x = 2$, then $6x = 12$ and $5x + 4 = 14$ and since $12 < 14$, $x = 2$ satisfies $6x < 5x + 4$. Two inequalities are equivalent if the same collection of numbers satisfies both inequalities.

Properties of inequalities

1. Adding the same expression to each side of an inequality gives an equivalent inequality (written $a < b \leftrightarrow a + c < b + c$ where $\leftrightarrow$ means equivalent).
2. Subtracting the same expression from each side of an inequality gives an equivalent inequality ($a < b \leftrightarrow a - c < b - c$).
3. Multiplying or dividing each side of an inequality by the same positive expression gives an equivalent inequality ($a < b \leftrightarrow ca < cb$ for $c > 0$).
4. Multiplying or dividing each side of an inequality by the same negative expression reverses the inequality ($a < b \leftrightarrow ca > cb$ for $c < 0$).
5. If both sides of an inequality have the same sign, inverting both sides of the inequality reverses the inequality.

$$0 < a < b \leftrightarrow 0 < \frac{1}{b} < \frac{1}{a}$$

$$a < b < 0 \leftrightarrow \frac{1}{b} < \frac{1}{a} < 0$$

6. If two inequalities are of the same type (both greater or both less), adding the respective sides gives the same type of inequality.

$$(a < b \text{ and } c < d, \text{ then } a + c < b + d)$$

Example 1: Find the values of x for which $5x-4<7x+2$.

Solution: Using property 2 subtract $5x + 2$ from each side, so $(5x - 4 < 7x + 2) \leftrightarrow -6 < 2x$. Now use property 3 and divide each side by 2, so $-6 < 2x \leftrightarrow -3 < x$.

Example 2. For what values of x is $\frac{3}{4}-\frac{5}{x}>\frac{7}{x}$?

Solution: $\frac{3}{4}-\frac{5}{x}>\frac{7}{x} \Rightarrow \frac{3}{4}>\frac{7}{x}+\frac{5}{x}=\frac{12}{x}$

Now be careful. Cross multiplying the inequality $\frac{3}{4} > \frac{12}{x}$ is essentially multiplying each side of the inequality by 4x.

So now you must consider two cases. If x is positive, so is 4x, and multiplying by 4x will preserve the inequality. However, if x is negative, so is 4x, and multiplying by 4x will reverse the inequality. Therefore, you must treat the two cases separately.

Case I: $x > 0$.

$$\frac{3}{4} > \frac{12}{x} \Rightarrow 3x > 48 \Rightarrow x > 16$$

So the positive solutions to the inequality are all numbers greater than 16.

Case II: $x < 0$

$$\frac{3}{4} > \frac{12}{x} \Rightarrow 3x < 48 \Rightarrow x < 16$$

So the negative solutions to the inequality are all negative numbers that are less than 16. Since every negative number is less than 16, every negative number is a solution.

Finally, by combining the two cases, we see that the solution set is $\{x \mid x<0 \text{ or } x > 16\}$.

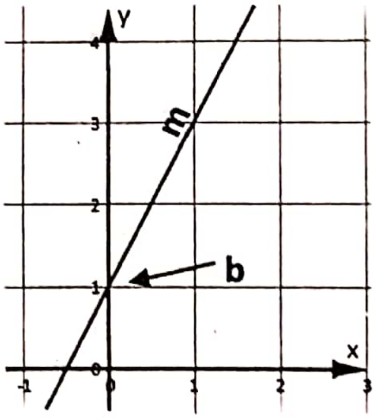
Example 3. For what value of x is $3\ |x+5| - 5 = 7$?

Solution: $3\ |x+5| - 5 = 7 \Rightarrow 3\ |x+5| = 12 \Rightarrow |x+5| = 4 \Rightarrow x+5 = 4 \text{ or } x+5 = -4$

$\Rightarrow x = -1 \text{ or } x = -9$

STRAIGHT LINE EQUATION SLOPE INTERCEPT FORM

The equation of a straight line on a graph is made up of y term, x term, and a number and are written in the form of y = mx + b. The slope of the line is known as the gradient and is represented by m in the equation. The point at which the line crosses the y-axis is the b in the equation. The equation of a straight line is usually written this way:



$y = mx + b$

Where

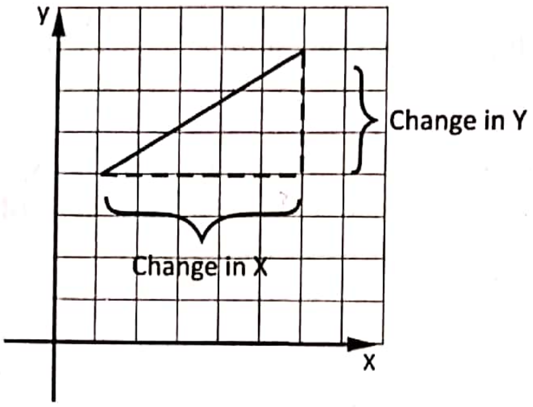
- y = how far up
- x = how far along
- **m** = Slope or Gradient (how steep the line is)
- **b or c** = the Y Intercept (where the line crosses the Y axis)

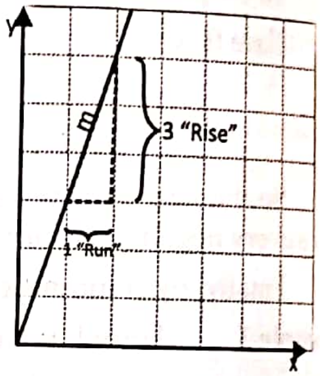
How to find "m" and "b"?

b is easy: just see where the line crosses the Y axis.

m (the Slope) needs some calculation:

$$m = \frac{\textit{Change in } Y}{\textit{Change in } X}$$





Knowing this we can work out the equation of a straight line:

Sometimes the words "rise" and "run" are used.

- Rise is how far up
- Run is how far along

And so the slope "m" is: $m = \frac{Rise}{Run}$

You might find that easier to remember

Example 1: Graph the equation, $y - 2x = 1$

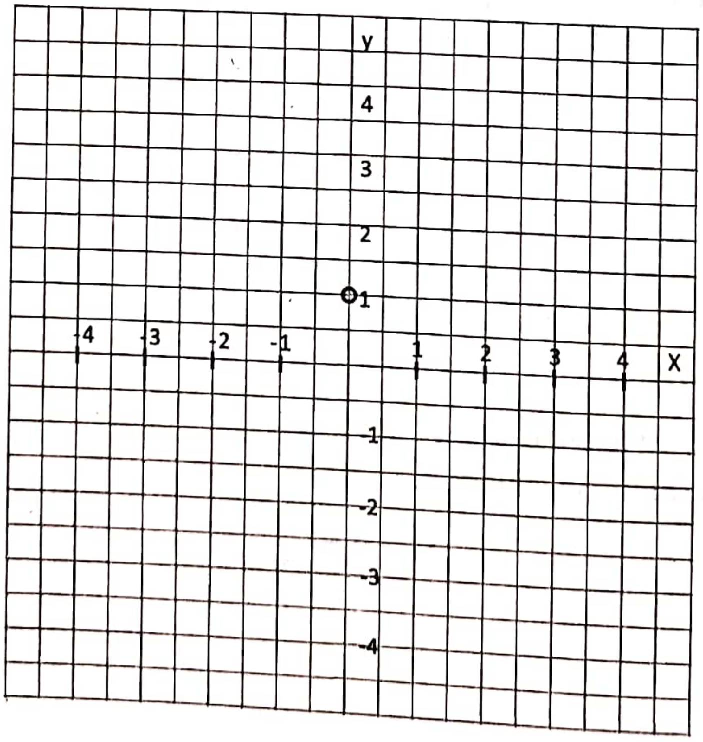
Rewrite in slope-intercept form $y = 2x+1$

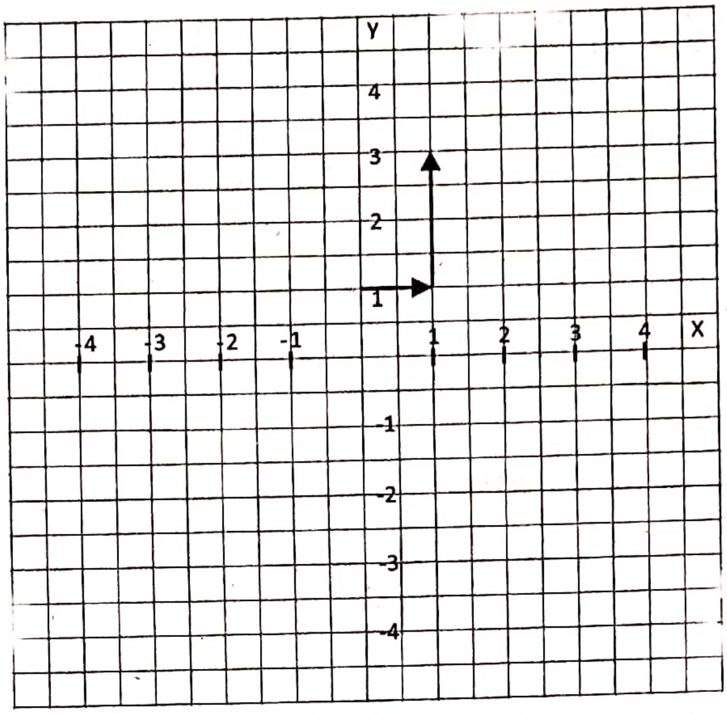
Identify the slope and the y-intercept

$m = 2$ and $b = 1$

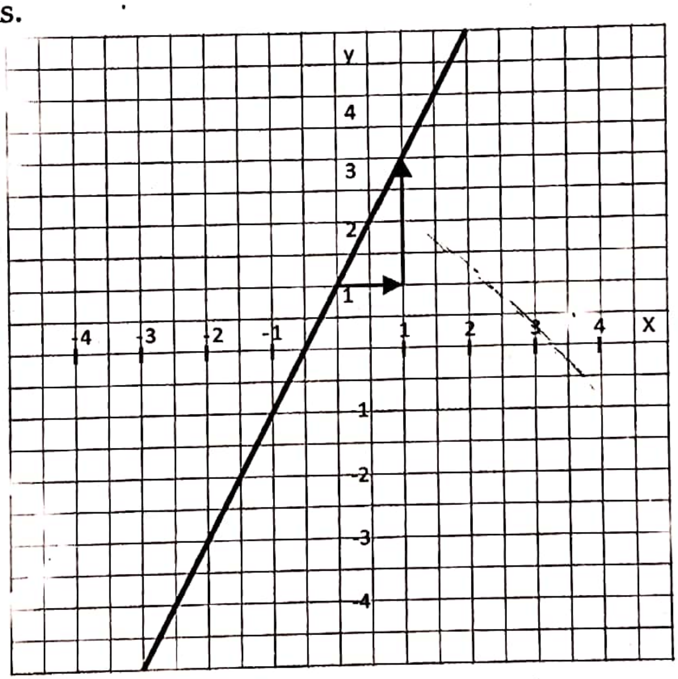
Plot the point corresponding to the y-intercept, (0,1)

The m-value, the slope, tells us that for each step to the right on the x-axis we move 2 steps upwards on the y-axis (since $m = 2$)





And once you have your second point you can just draw a line through the two points and extend it in both directions.



You can check to see that the line you've drawn is the correct one by substituting the coordinates of the second point into the original equation. If the equation holds true than the second point is correct.

Our second point = (1, 3)

$$y-2x=1$$

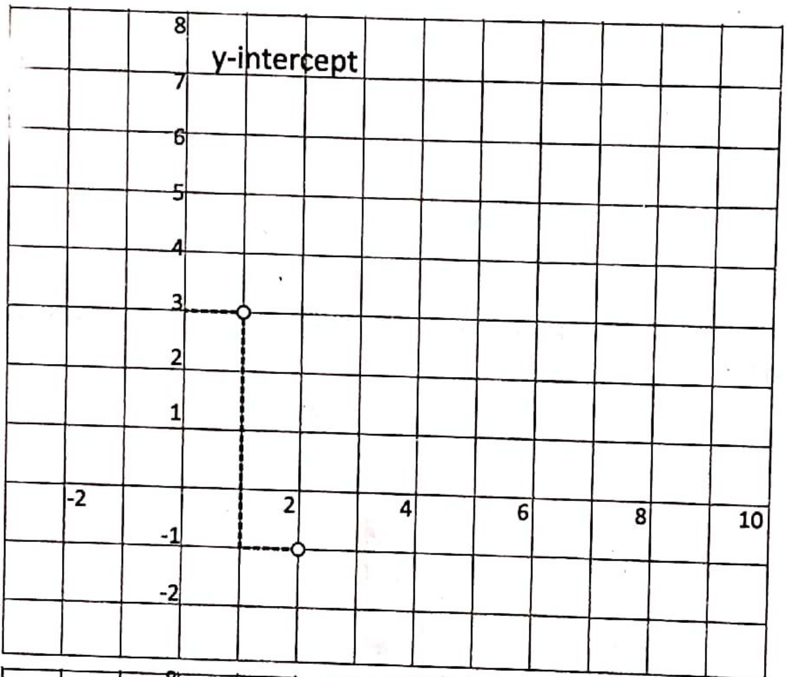
$$3-2=1$$

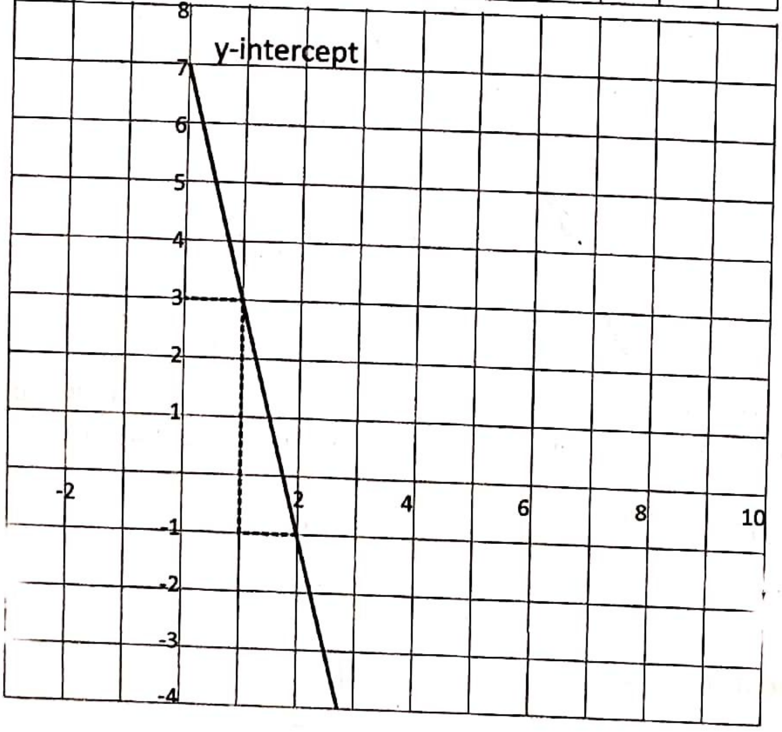
Example 2: Graph the equation $4x + y = 7$.

Unfortunately, this equation is not given to us in slope-intercept form. However, we can solve the equation for y and manipulate it into this form.

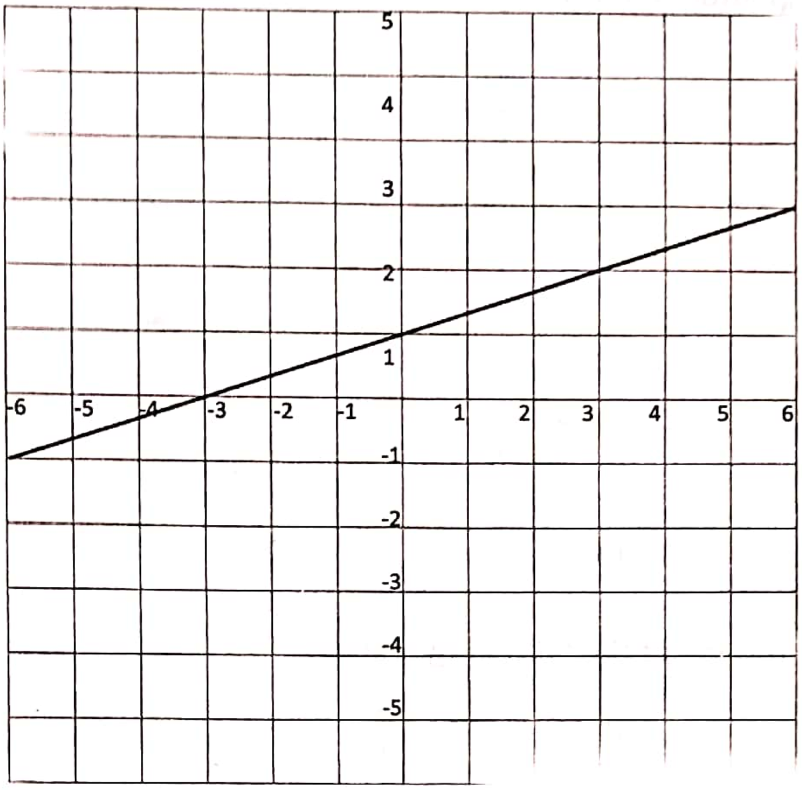
$4x + y = 7$	
$4x + y - 4x = -4x$	subtract 4x from each side
$y = 7 - 4x$	simplify
$y = -4x + 7$	switch the 7 and 4x, but keep the appropriate signs

Now we can see that the slope of this equation is -4, also known as -4/1 when written as a fraction, and the y-intercept is 7.





Example 3: Find the equation of the line in slope-intercept form.



Let's look carefully at the graph and see if we can find any important information....

we can see that the y-intercept is 1. We can also find the slope by counting the change in the vertical and horizontal directions between the two points shown.

From the point (0, 1) to (3, 2), our line is has a "rise" of 1 and a "run" of 3. Slope is rise over run, so that gives us a slope of ⅓.

Now using that information, we know these things:

$b = 1$ and $m = \frac{1}{3}$

All we need to do is plug them into our equation, $y = mx + b$.

$y = \frac{1}{3}x + 1$

EXERCISE

Answers Bold and Underline:

1. **For the straight line $y = -2x + 3$, what are: (a) the slope and (b) the y-intercept?**

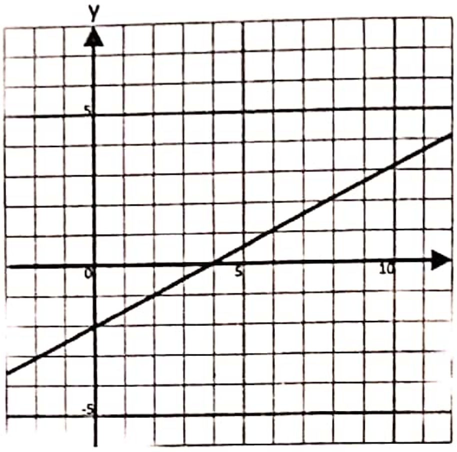
(a) Slope = 2, y-intercept = -3

(b) **Slope = -2, y-intercept = 3**

(c) Slope = 3, y-intercept = -2

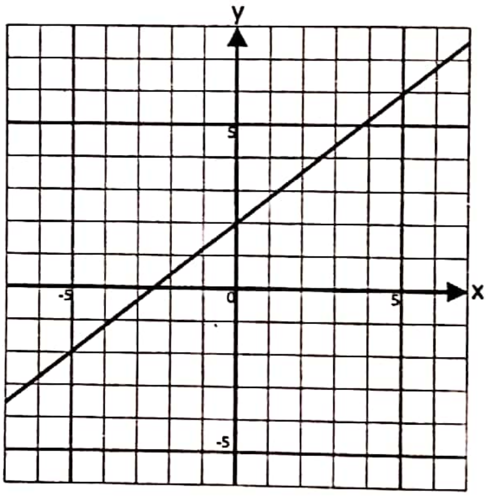
(d) Slope = -3, y-intercept = 2

2. **What is the equation of the straight line shown in the diagram?**



(a) y = 2x – 2 **(b)** y = -2x + ½ **(c)** y = -(½)x + 2 **(d)** y = (½)x - 2

3. **What is the equation of the following line?**



(a) y = -1.33x + 2

(b) y = 1.33x + 2

(c) y = -0.75x + 2

(d) **y = 0.75x + 2**

CHAPTER 28

SETS AND SET THEORY

A **set** is a collection of objects that have something in common or follow a rule. The objects in the set are called its **elements**. Set notation uses curly braces { } with elements separated by commas. So the set of outwear for Salman would be listed as follows:

A = {coat, hat, scarf, gloves, boots},

Example 1: A = {coat, hat, scarf, gloves, boots}, where A is the name of the set, and the braces indicate that the objects written between them belong to the set. Every object in a set is unique: The same object cannot be included in the set more than once. Let's look at some more examples of sets.

Example 2: What is the set of all fingers?

Solution: P = {thumb, index, middle, ring, little}

Example 3: What is the set of all even whole numbers between 0 and 10?

Solution: Q = {2, 4, 6, 8}

Note: The use of the word' between' means that the range of numbers given is not inclusive. As a result, the numbers 0 and 10 are not listed as elements in this set.

Notations in Sets

To learn about sets we shall use some accepted notations for the familiar sets of numbers.

The following conventions are used with sets:

- **Capital letters** are used to denote sets.
- **Lowercase letters** are used to denote elements of sets.
- **Curly braces** { } denote a list of elements in a set.

These are the different notations in sets generally required while solving various types of problems on sets.

∈	Belongs to
∉	Does not belongs to
: or |	Such that
∅	Null set or empty set
n(a)	Cardinal number of the set A
∪	Union of two sets
∩	Intersection of two sets

N	Set of natural numbers = {1, 2, 3, ...}
W	Set of whole numbers = {0, 1, 2, 3, ...}
I or Z	Set of integers = {........., -2, -1, 0, 1, 2, ...}
Z+	Set of all positive integers
Q	Set of all rational numbers
Q+	Set of all positive rational numbers

Representation of Set

In representation of a set the following three methods are commonly used:

1. Statement form method
2. Roster or tabular form method
3. Rule or set builder form method

Statement Form

In statement form method, well-defined description of the elements of the set is given and the same are enclosed in curly brackets.

Examples:

1. Set of all even numbers less than 10 can be written in Statement form as

A={Even numbers less than 10}

2. Set of all vowels can be written as

A={Set of all vowels}

Roster or Tabular Form

In Roster form, elements of the set are listed within the pair of brackets { } and are separated by commas.

Examples:

(i) Let N denote the set of first five natural numbers.

Therefore, N = {1, 2, 3, 4, 5} → Roster Form

(ii) The set of all vowels of the English alphabet.

Therefore, V = {a, e, i, o, u} → Roster Form

(iii) The set of all odd numbers less than 9.

Therefore, X = {1, 3, 5, 7} → Roster Form

Set Builder Form

In this form of representation of a set, the element of the set is described by using a symbol 'x' or any other variable followed by a colon The symbol ':' or '|' is used to denote such that and then we write the property possessed by the elements of the set and enclose the whole description in braces. In this, the colon stands for 'such that' and braces stand for 'set of all'.

Example:

(i) Let P is a set of counting numbers greater than 12; the set P in set-builder form is written as:

P = {x : x is a counting number and greater than 12}

or

P = {x | x is a counting number and greater than 12}

This will be read as, 'P is the set of elements x such that x is a counting number and is greater than 12'.

(ii) Let A denote the set of even numbers between 6 and 14. It can be written in the set builder form as;

A = {x | x is an even number, $6 < x < 14$}

or

A = {x : x ∈ P, $6 < x < 14$ and P is an even number}

(iii) If X = {4, 5, 6, 7} . This is expressed in roster form. Let us express in set builder form.

X = {x : x is a natural number and $3 < x < 8$}

Properties of sets

The two basic properties to represent a set are explained below using various examples.

1. The change in order of writing the elements does not make any changes in the set.

In other words the order in which the elements of a set are written is not important. Thus, the set {a, b, c} can also be written as {a, c, b} or {b, c, a} or {b, a, c} or {c, a, b} or {c, b, a}.

Example:

Set A = {4, 6, 7, 8, 9} is same as set A = {8, 4, 9, 7, 6}

i.e. {4, 6, 7, 8, 9} = {8, 4, 9, 7, 6}

Similarly, {w, x, y, z} = {x, z, w, y} = {z, w, x, y} and so on.

2. If one or many elements of a set are repeated, the set remains the same.

In other words. the elements of a set should be distinct. So, if any element of a set is repeated number of times in the set, we consider it as a single element. Thus, {1, 1, 2, 2, 3, 3, 4, 4, 4} = {1, 2, 3, 4}

The set of letters in the word 'GOOGLE' = {G, O, G, L, E}

In general, the elements of a set are not repeated. Thus,

(i) if T is a set of letters of the word 'moon': then T = {m, o, n},

There are two O's in the word 'moon' but it will be written in the set only once.

(ii) if U = {letters of the word 'COMMITTEE'}; then U = {C, O, M, I, T, E}

Types of Sets

The different types of sets are explained below with examples.

Empty Set or Null Set: A set which does not contain any element is called an empty set, or the null set or the void set and it is denoted by ∅ and is read as **phi**. In roster form, ∅ is denoted by {}.

Example:

(a) The set of whole numbers less than 0.

Clearly there is no whole number less than 0.

Therefore, it is an empty set.

$N = \{x : x \in W, x < 0\}$

(b) Let A = {x : x is a natural number, 2 < x < 3}

Here A is an empty set because there is no natural number between 2 and 3.

Note: $\emptyset \neq \{0\}$ ∴ has no element.

{0} is a set which has one element 0.

The cardinal number of an empty set, i.e., $n(\emptyset) = 0$

Singleton Set or Unit Set: A set which contains only one element is called a singleton set.

Examples:

- A = {x : x is neither prime nor composite}

It is a singleton set containing one element, i.e., 1.

- B = {x : x is a whole number, x < 1}

This set contains only one element 0 and is a singleton set.

Finite Set: A set which contains a definite number of elements is called a finite set. Empty set is also called a finite set.

Examples:

- The set of all colours in the rainbow .

R = {V, I, B, G, Y, O, R,}

- Set of Prime numbers up to 97

P = {2, 3, 5, 7, 11, 13, 17, ... 97}

Infinite Set: The set whose elements cannot be listed, i.e. set containing never-ending elements is called an infinite set.

Examples:

- Set of all points in a plane
- Set of all prime numbers P = {2, 3, 5, 7, ...}

Cardinal Number of a Set: The number of distinct elements in a given set A is called the cardinal number of A. It is denoted by n(a).

Example:

- A {x : x ∈N, x < 5}

A = {1, 2, 3, 4}

Therefore, n(a) = 4

- B = Set of letters in the word ALGEBRA

B = {A, L, G, E, B, R}

Therefore, n(b) = 6

Equivalent Sets: Two sets A and B are said to be equivalent if their cardinal number is same, i.e., n(a) = n(b). The symbol for denoting an equivalent set is '↔'.

Examples:

A = {1, 2, 3} Here n(a) = 3

B = {p, q, r} Here n(**b**) = 3

Therefore, A ↔ B

Equal sets: Two sets A and B are said to be equal if they contain the **same elements**. The symbol for denoting an equal set is '='.

Examples:

A = {p, q, r, s}

B = {p, s, r, q}

Therefore, A = B

SET OPERATIONS

We have various operations on sets like union of sets, intersection of sets, difference of two sets, and complement of a set.

Union of Sets

Union of two given sets is the smallest combined set which contains all the elements of both the sets without repetition. The symbol for denoting union of sets is '∪'.

Union of two sets A and B is the set of all elements which are in A or in B, or both in A and B. In symbol, we can write, A∪B for the union of two sets A and B. "A ∪ B" is read as "A union B"

Example:

Let A = {3, 4, 5, 6}, B = {6, 7, 8} and C = {8, 9, 7}.

Then find A∪B, B∪C and A∪C.

$$A \cup B = \{3, 4, 5, 6, 7, 8\}$$

$$B \cup C = \{6, 7, 8, 9\}$$

$$A \cup C = \{3, 4, 5, 6, 7, 8, 9\}$$

Intersection of Sets:

A set can be formed by using all the **common elements** of two given sets. Such a collection is called the intersection of the given sets. The symbol for denoting intersection of sets is '∩'.

Example:

Let A = {3, 4, 5, 6}, B = {5, 6, 7}, C = {7, 8, 9}

Then find A ∩ B, B ∩ C and A ∩ C.

$A \cap B = \{5, 6\}$, since $5 \in A, 5 \in B, 6 \in A, 6 \in B$

$B \cap C = \{7\}$, since $7 \in B, 7 \in A$

$A \cap C = \phi$, since no elements are common.

Complement of Set

Let U be a universal set and A be any subset of U, then the elements of U which are not in A ,i.e. U - A is the complement of A w.r.t. U is written as $A' = U - A = A^c$. [$A^c = A' = \{x : x \in U$ and $x \notin A\}$

For example U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} and A = { 1, 2, 4, 6, 9}

Then, $A' = U - A = \{0, 3, 5, 7, 8\}$

Subset

If A and B are two sets, and every element of set A is also an element of set B, then A is called a subset of B and we write it as **$A \subseteq B$ or $B \supseteq A$.**

Remember

- Every set is a subset of itself, i.e., $A \subseteq A$, $B \subseteq B$.
- Empty set is a subset of every set.
- Symbol '$\subseteq$' is used to denote 'is a subset of' or 'is contained in'.
- $A \subseteq B$ means A is a subset of B or A is contained in B.

Example:

Let $A = \{2, 4, 6\}$ and $B = \{6, 4, 8, 2\}$

Here A is a subset of B. $A \subseteq B$

Since, all the elements of set A are contained in set B.

But B is not the subset of A. Since, all the elements of set B are not contained in set A.

Super Set

Whenever a set A is a subset of set B, it means, all elements of set A are present in set B. But set B has at least one element that is not present in A. Set B is a superset of A and we write $B \supseteq A$. Symbol $\supseteq$ is used to denote 'is a super set of.

Example:

$A = \{a, e, i, o, u\}$ and $B = \{a, b, c, \ldots\ldots\ldots\ldots, z\}$

Here $A \subseteq B$, i.e. A is a subset of B but $B \supseteq A$, i.e., B is a super set of A.

Proper Subset

If A and B are two sets, then A is called the proper subset of B if $A \subseteq B$ but $B \supseteq A$, i.e. $A \neq B$. The symbol '$\subset$' is used to denote proper subset. Symbolically, we write $A \subset B$.

Example:

$A = \{1, 2, 3, 4\}$ Here n(**a**) = 4

$B = \{1, 2, 3, 4, 5\}$ Here n(**b**) = 5

We observe that all the elements of A are present in B but the element '5' of B is not present in A. So, we say that A is a proper subset of B. Symbolically, we write it as $A \subset B$.

Remember

- No set is a proper subset of itself.
- Null set or $\emptyset$ is a proper subset of every set.
- If a set contains 'n' elements, then the number of subsets of the set is 2^n.
- If a set contains 'n' elements, then the number of proper subsets of the set is $2^n - 1$.

If $A = \{p, q\}$ the proper subsets of A are $[\{\ \}, \{p\}, \{q\}]$

$\Rightarrow$ Number of proper subsets of A are $3 = 2^2 - 1 = 4 - 1$.

Example: If A {1, 3, 5}, then write all the possible subsets of A. Find their numbers.

Solution:

The subset of A containing no elements - { }

The subset of A containing one element each - {1} {3} {5}

The subset of A containing two elements each - {1, 3} {1, 5} {3, 5}

The subset of A containing three elements - {1, 3, 5)

Therefore, all possible subsets of A are { }, {1}, {3}, {5}, {1, 3}, {3, 5}, {1, 3, 5}

Therefore, number of all possible subsets of A is 8 which is equal 2^3.

Proper subsets are = { }, {1}, {3}, {5}, {1, 3}, {3, 5}

Number of proper subsets are $7 = 8 - 1 = 2^3 - 1$

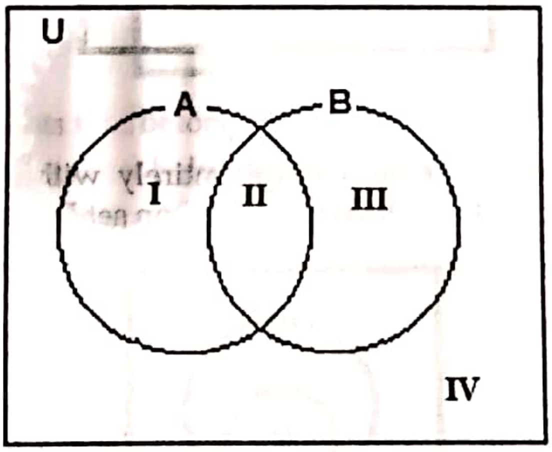
VENN DIAGRAMS

In mathematics, we can use the graphs and diagrams to solve some problems in geometry as well as in algebra. To follow this procedure, we can show some relations in set theory with the help of diagram, which is called as the **Venn diagram**. It is also known as **set diagram**. Venn diagrams are named so in the name of its founder John Venn.

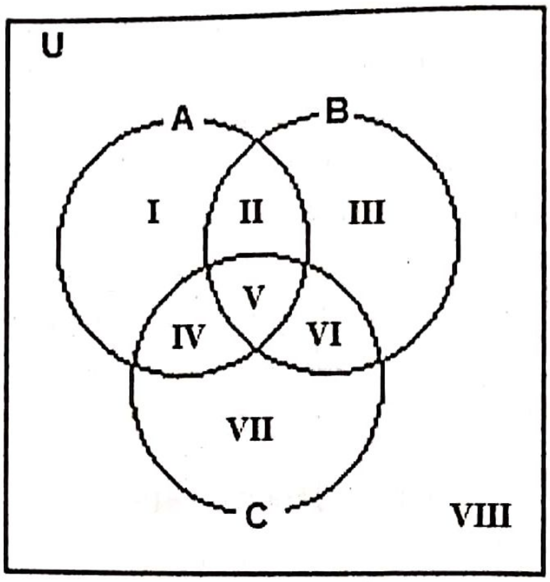
A Venn diagram is a pictorial representation of sets by set of points in the plane. The universal set U is represented pictorially by interior of a rectangle and the other sets are represented by closed figures, viz. Circles or ellipses or small rectangles or some curved figures lying within the rectangle.

Types of Venn Diagrams

we will discuss Venn diagrams involving two sets and three sets. Each distinct region within a Venn diagram will be identified by using Roman Numerals for the numbers 1 through 4 (in the two-set diagram) and the numbers 1 through 8 (in the three-set diagram).



Two-Set Venn Diagram

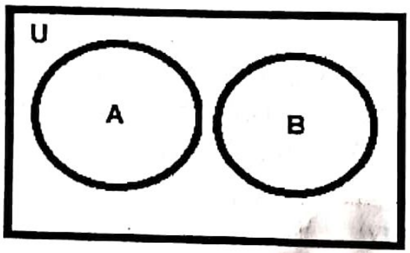


Three-Set Venn Diagram

Relationships in Two-Set Venn diagrams

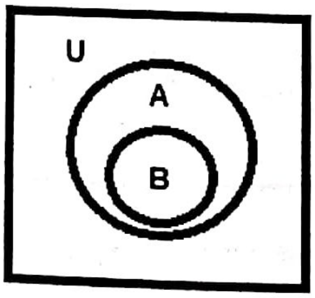
Disjoint Sets

Disjoint sets are represented by circles that do not overlap. That is, no elements of set A are elements of set B and vice versa.



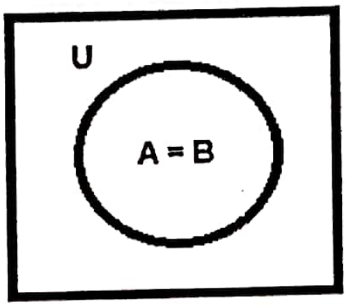
Proper Subsets

Proper subsets are represented by a circle entirely within a larger circle. That is, all elements of set B are elements of set A, which is larger than set B.



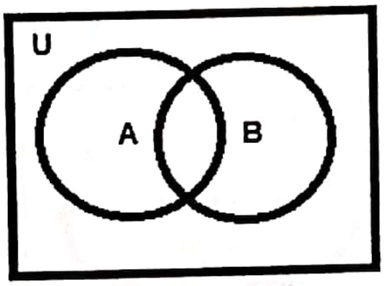
Equal Sets

If A = B, then set A contains exactly the same elements as set B.

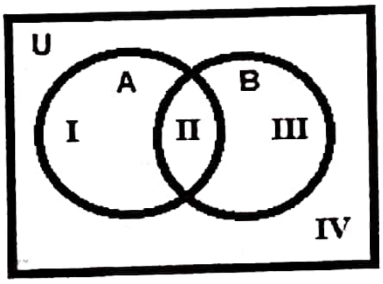


Sets with common elements

If set A and set B have at least one element in common, then the circles representing the sets must overlap.



The above Venn diagram can then be changed by numbering each of the regions as follows:



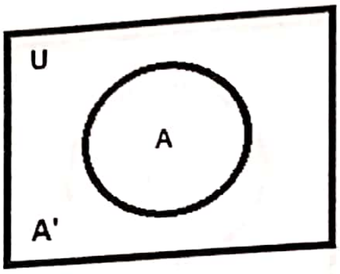
Region I — The elements that belong to set A, but not to set B.
Region II — The elements that belong to set A and to set B.
Region III — The elements that belong to set B, but not to set A.
Region IV — The elements that belong to the universal set U that are not in sets A or B.

The Complement of a Set

The complement of set A, symbolized by A', is the set of all elements in the universal set that are **NOT** in set A. In Set-Builder notation this idea can be expressed as:

$$A' = \{x \mid x \in U \text{ and } x \in A\}$$

This concept is expressed as follows where the area A' lies outside set A but within the universal set U.

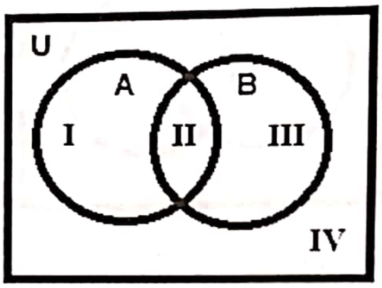


The Intersection of Sets

The intersection of sets A and B, symbolized by $A \cap B$ is a listing of elements that are common to both set A **and** set B. In Set-Builder notation this idea can be expressed as:

$$A \cap B = \{x \mid x \in A \, and \, x \in B\}$$

This concept is expressed as follows where the area $A \cap B$ is the Region II.

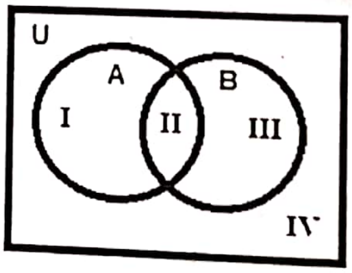


The Union of Sets

The union of sets A and B, symbolized by $A \cup B$, is a listing of elements of set A plus the elements in set B that are not already listed. In Set-Builder notation this idea can be expressed as

$$A \cup B = \{x \mid x \in A \, or \, x \in B\} .$$

Remember, in set theory we are using **inclusive or,** so this means the set $A \cup B$ contains all of the elements that are in set A plus the elements in set B that are not already listed. Pictorially, this concept is expressed as follows where the area $A \cup B$ consists of the Regions I, II, and III.



SELF-EXERCISES

1. List out the elements of the set "The letters of the word Mississippi"
2. List out the elements of the set "Months of the year"
3. Write a verbal description of the set {3, 6, 9}
4. Write a verbal description of the set {a, i, e, o, u}

5. Is {1, 3, 5} a subset of the set of odd integers?
6. Is {A, B, C} a subset of the set of letters of the alphabet?

For problems 7-12, consider the sets below, and indicate if each statement is true or false.

A = {1, 2, 3, 4, 5} B = {1, 3, 5} C = {4, 6}U = {numbers from 0 to 10}

7. $3 \in B$
8. $5 \in C$
9. $B \subset A$
10. $C \subset A$
11. $C \subset B$
12. $C \subset D$

Using the sets from above, and treating U as the Universal set, find each of the following:

13. $A \cup B$
14. $A \cup C$
15. $A \cap C$
16. $B \cap C$
17. A^c
18. B^c

Q19 to 22. Let D = {b, a, c, k}, E = {t, a, s, k}, F = {b, a, t, h}. Using these sets, find the following:

19. $D^c \cap E$
20. $F^c \cap D$
21. $(D \cap E) \cup F$
22. $D \cap (E \cup F)$
23. List the complement of each set A, for the given Universal set, S.
    - **(a)** S = the set of whole numbers less than 25
      A = {2, 3, 5, 10, 15, 20}
    - **(b)** S = the set of letters in the alphabet
      A = the set of consonants
24. The Swiss embassy has 65 employees. Of these employees, 47 speak German, 35 Speak Italian, and 20 speak both. How many speak neither language? Illustrate this situation with a Venn diagram.

25. Find the union of each of the following pairs of sets.

(a) $A = \{2, 4, 6\}$ $B = \{1, 2, 3\}$

(b) $P = \{a, e, i, o, u\}$ $Q = \{a, b, c, d\}$

(c) $X = \{x : n \in N, x = 2n, n < 4\}$ $Y = \{x :$ x is an even number less than $10\}$

25. Find the intersection of each of the following pairs of sets.

(a) $A = \{1, 4, 9, 16\}$ $B = \{3, 6, 9, 12\}$

(b) $C = \{p, q, r, s\}$ $D = \{a, b\}$

(c) $P = \{x : n \in N, x = 3n \;\; n < 3\}$ $Q = \{x : x \in N \;\; x < 7\}$

26. If $P = \{1, 2, 3\}$ $Q = \{2, 3, 4\}$ $R = \{3, 4, 5\}$ $S = \{4, 5, 6\}$, find

(a) $P \cup Q$ **(b)** $P \cup R$ **(c)** $Q \cup R$

(e) $P \cup Q \cup R$ **(f)** $P \cup Q \cup S$ **(g)** $Q \cup R \cup S$

(h) $P \cap Q$ **(i)** $P \cap R$ **(j)** $Q \cap R$

(l) $P \cap Q \cap R$ **(m)** $P \cap Q \cap S$ **(n)** $Q \cap R \cap S$

27. If $A = \{a, b, c, d\}$ $B = \{b, c, d, e\}$ $C = \{c, d, e, f\}$ $D = \{d, e, f, g\}$, find

(a) $A - B$ **(b)** $B - C$ **(c)** $C - D$

28. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ $A = \{1, 2, 4, 6, 8, 10\}$ and $B = \{1, 3, 5, 7, 8, 9\}$ Find:

(a) A' **(b)** B' **(c)** $A' \cup B'$ **(d)** $A' \cap B'$ **(e)** $(A \cup B)'$

Also show $(A \cup B)' = A' \cap B'$.

29. Let $P = \{1, 3, 5, 7\}$ $Q = \{3, 7, 9, 11\}$ $R = \{1, 5, 8, 11\}$, then verify the following.

(a) $P \cup Q = Q \cup P$

(b) $(P \cup Q) \cup R = P \cup (Q \cup R)$

(c) $P \cap Q = Q \cap P$

(d) $(P \cap Q) \cap R = P \cap (Q \cap R)$

(e) $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$

(f) $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$

30. Write the following sets in the set builder form.

(a) $A = \{2, 4, 6, 8\}$ **(b)** $B = \{3, 9, 27, 81\}$ **(c)** $C = \{1, 4, 9, 16, 25\}$

31. Write the following sets in the roster form.

(a) $A = \{x : x \in W, x \leq 5\}$ **(b)** $B = \{x : x \in I, -3 < x < 3)$

(c) $C = \{x :$ x is divisible by $12\}$

32. Which of the following are the examples of an empty set?

(a) The set of even natural numbers divisible by 3.

(b) The set of all prime numbers divisible by 2.

(c) $\{x : x \in N, 5 < x < 6\}$

(d) The set of odd natural numbers divisible by 2.

(e) $B = \{O\}$

(f) $C = \{\ \}$

(g) $D = \{x : x \in R, x2 = -1\}$

33. Classify the following as finite and infinite sets.

(a) The set of days in a week

(b) $A = \{x : x \in N\ x > 1\}$

(c) $B = \{x : x$ is an even prime number$\}$

(d) $C = \{x : x$ is a multiple of $5\}$

(e) $D = \{x : x$ is a factor of $30\}$

34. From the sets given below, identify the equal sets.

$A = \{3, 5, 9, 11\}$ $Q = \{m, s, t\}$
$B = \{8, 9, 1, 13\}$ $R = \{o, p, a, z\}$
$C = \{-3, 3\}$ $T = \{1, 8, 9, 13\}$
$D = \{s, t, m\}$ $M = \{3, -3\}$
$P = \{9, 3, 5, 11\}$ $X = \{a, o, z, p\}$

35. Find the cardinal number of the following sets.

(a) $A = \{x : x \in I, 2 < x < 7\}$

(b) $B = \{x : n \in N, x = n2, n < 3\}$

(c) The set of months in a year

(d) $C = \{x : x \in Z+, x < 100\}$

(e) $D = \{x : x = n3, n \in W, n < 5\}$

CHAPTER 29

WORD PROBLEMS

TRANSLATING WORD PROBLEMS

The hardest thing about doing word problems is taking the English words and translating them into mathematics. Usually, once you get the math equation, you are fine; the actual math involved is often simple. However, figuring out the actual equation can seem tough.

The **first step** to effectively translating and solving word problems is to read the problem entirely. Do not start trying to solve anything when you have only read half a sentence. Try first to get a feel for the whole problem; try first to see what information you have, and what you still need

The **second step** is to work in an organized manner. Figure out what you need but do not have, and name things. Pick variables to stand for the unknowns, clearly labelling these variables with what they stand for.

The **third step** is to look for "**key words**". Certain words indicate certain mathematical operations. Below is a partial list.

Addition	increased by, more than, combined, together, total of sum, plus, added to, altogether, both, in all...
Subtraction	decreased by, minus, less, difference between/of less than, fewer than, how many more left less than, remaining, take away...
Multiplication	Of, times, multiplied by, product of, increased/decreased by a factor of (this type can involve both addition or subtraction and multiplication!), group of...
Division	per, a, out of, ratio of, quotient of, percent (divide by 100)...
Equals	is, are, was, were, will be, gives, yields, sold for, balance, same as, equals...

Warning: The "less than" construction is backwards in the English from what it is in the math. If you need to translate "1.5 less than x", the temptation is to write "$1.5 - x$". Do not do this! You can see how this is wrong by using this construction in a "real world" situation: Consider the statement, "He makes $1.50 an hour less than me." You do not figure his wage by subtracting your wage from $1.50. Instead, you subtract $1.50 from your wage. So remember; **the "less than" construction is backwards.**

Note that order is important in the "quotient/ratio of" and "difference between/of" constructions. If a problems says "the ratio of x and y", it means, "x divided by y", not "y divided by x". If the problem says "the difference of x and y", it means "$x - y$", not "$y - x$".

COMMON MATH SYMBOLS

Symbols save time and space when writing. Here are the **most common** mathematical symbols:

Symbol	**Symbol Name**	**Meaning / definition**	**Example**
$=$	equals sign	Equality	$5 = 2+3$
$\neq$	not equal sign	Inequality	$5 \neq 4$
$>$	strict inequality	greater than	$5 > 4$
$<$	strict inequality	less than	$4 < 5$
$\geq$	Inequality	greater than or equal to	$5 \geq 4$
$\leq$	Inequality	less than or equal to	$4 \leq 5$
( )	Parentheses	calculate expression inside first	$2 \times (3+5) = 16$
[ ]	Brackets	calculate expression inside first	$[(1+2)*(1+5)] = 18$
$\cdot$	multiplication dot	Multiplication	$2 \cdot 3 = 6$
/	division slash	Division	$6 / 2 = 3$
a^b	Power	Exponent	$2^3 = 8$
a^b	Caret	Exponent	2 ^ 3 = 8
$\sqrt{a}$	square root	$\sqrt{a} \cdot \sqrt{a} = a$	$\sqrt{9} = \pm 3$
$\sqrt[3]{a}$	cube root	$\sqrt[3]{a} \cdot \sqrt[3]{a} \cdot \sqrt[3]{a} = a$	$\sqrt[3]{8} = 2$
%	Percent	1% = 1/100	10% × 30 = 3
$\angle$	Angle	formed by two rays	$\angle ABC = 30^\circ$
$\measuredangle$	measured angle	Shows magnitude of angle	$\measuredangle ABC = 30^\circ$
∟	right angle	$= 90^\circ$	$\alpha = 90^\circ$
$\therefore$	Therefore	In consequence of	
$\because$	because / since		

AGE PROBLEMS

The majority of age problems include the present ages of two or more persons, in addition to their ages at particular past and/or future times. The key to solving these problems is to translate the verbiage into equations using subscripted variables.

Example 1: Ahmad is twice as old as his friend Hassan. Hassan is 5 years older than Fatima. In 5 years, Ahmad will be three times as old as Fatima. How old is Hassan now?

Solution: **Step 1:** Set up a table.

	Age now	Age in 5 years
Ahmad		
Hassan		
Fatima		

Step 2: Fill in the table with information given in the question. Let age of Fatima is x. Hassan is 5 years older than Fatima. Ahmad is twice as old as his friend Hassan.

	Age now	Age in 5 years
Ahmad	$2(x+5)$	$2(x+5)+5=2x+10$
Hassan	$x+5$	$x+5+5=x+10$
Fatima	x	$x+5$

Write the new relationship in an equation using the ages in 5 years.

In 5 years, Ahmad will be three times as old as Fatima will be.

$$2x+10=3(x+5)$$
$$2x+10=3x+15$$

Isolate variable x

$$x=5$$

Answer: Hassan is now 5 years old.

Example 2: Ahmad 's father is 5 times older than Ahmad and Ahmad is twice as old as his sister Fatima . In two years' time, the sum of their ages will be 58. How old is Ahmad now?

***Solution:* Step 1:** Set up a table.

	Age now	Age in 2 years
Ahmad 's father		
Ahmad		
Fatima		

Step 2: Fill in the table with information given in the question.

Let x be Ahmad's age now. Add 2 to get the ages in 2 years.

	Age now	Age in 2 years
Ahmad 's father	$5x$	$5x+2$
Ahmad	X	$x+2$
Fatima	$\frac{1}{2}x$	$\frac{1}{2}x+2$

Write the new relationship in an equation using the ages in 2 years. In two years' time, the sum of their ages will be 58.

$$5x+2+x+2+\frac{1}{2}x+2=58$$
$$5x+x+\frac{1}{2}x=58-2-2-2$$
$$6\frac{1}{2}x=52$$
$$x=8$$

Answer: Ahmad is now 8 years old.

Example 3: Yasmeen is twice as old as Batool. Three years from now, the sum of their ages will be 42. How old is Yasmeen?

We'll let Batool's age be x. We can set up a chart:

	Now	In 3 years
Yasmeen	$2x$	$2x + 3$
Batool	X	$x + 3$

The sum of their ages in 3 years will be 42, so we have:

$$(2x + 3) + (x + 3) = 42$$
$$3x + 6 = 42$$
$$3x = 36$$
$$x = 12$$

If Batool is 12, Yasmeen is 24; in three years, they will be 15 and 27, and the sum of their ages will be 42.

EXERCISE -I

Question: 1

In three more years, Omar's grandfather will be six times as old as Omar was last year. When Omar's present age is added to his grandfather's present age, the total is 68. How old is each one now?

Label Omar's present age as "m" and his grandfather's present age as "g". Then $m + g = 68$. Omar's age "last year" was $m - 1$. His grandfather's age "in three more years" will be $g + 3$. The grandfather's "age three years from now" is six times Omar's "age last year" or, in math:

$$g + 3 = 6(m - 1)$$

This gives me two equations with two variables:

$$m + g = 68$$
$$g + 3 = 6(m - 1)$$

Solving the first equation, $m = 68 - g$. (Note: its okay to solve for "$g = 68 - m$", too. The problem will work out a bit differently in the middle, but the answer will be the same at the end.) Plug "$68 - g$" into the second equation in place of "m":

$$g + 3 = 6m - 6$$
$$g + 3 = 6(68 - g) - 6$$
$$g + 3 = 408 - 6g - 6$$
$$g + 3 = 402 - 6g$$
$$g + 6g = 402 - 3$$
$$7g = 399$$
$$g = 57$$

Since "g" stands for the grandfather's current age, then the grandfather is 57 years old. Since $m + g = 68$, then $m = 11$, and Omar is presently eleven years old.

Question: 2

One-half of Hafsa's age two years from now plus one-third of her age three years ago is twenty years. How old is she now?

This problem refers to Hafsa's age two years in the future and three years in the past. So pick a variable and label everything clearly:

age now: H

age two years from now: $H + 2$

age three years ago: $H - 3$

one-half of age two years from now: $(^1/_2)(H + 2) = {}^H/_2 + 1$

one-third of age three years ago: $(^1/_3)(H - 3) = {}^H/_3 - 1$

The sum of these two numbers is twenty, so add them and set this equal to 20:

$$^H/_2 + 1 + {}^H/_3 - 1 = 20$$

$$^H/_2 + {}^H/_3 = 20$$

$$3H + 2H = 120$$

$$5H = 120$$

$$H = 24$$

Hafsa is 24 years old.

Question: 3

Babar is 4 years older than Wasim, and Qasim is twice as old as Babar. Three years ago, the sum of their ages was 35. How old is each now?

x = Wasim's age now

$x + 4$ = Babar's age now {Babar is 4 years older than Wasim}

$2(x + 4) = 2x + 8$ = Qasim's age now {Qasim is twice as old as Babar, used distributive property}

$x - 3$ = Wasim's age 3 years ago {subtracted 3 from x}

$x + 1$ = Babar's age 3 years ago {subtracted 3 from $x + 4$}

$2x + 5$ = Qasim's age 3 years ago {subtracted 3 from $2x + 8$}

$(x - 3) + (x + 1) + (2x + 5) = 35$ {sum of ages, 3 years ago, was 35}

$x - 3 + x + 1 + 2x + 5 = 35$ {took out parentheses}

$4x + 3 = 35$ {combined like terms}

$4x = 32$ {subtracted 3 from both sides}

$x = 8$ = Wasim now {divided both sides by 4}

$x + 4 = 12$ = Babar now {substituted 8, in for x, into $x + 4$}

$2(x + 4) = 24$ = Qasim now {substituted 8, in for x, into $2(x + 4)$}

Wasim is 8 now, Babar is 12 now and Qasim is 24 now

Question: 4

An eagle is 4 times as old as a falcon. Three years ago, the eagle was 7 times as old as the falcon. Find the present age of each bird.

x = falcon's age now

$4x$ = eagle's age now {the eagle is 4 times as old as falcon}

$x - 3$ = falcon's age 3 years ago

$4x - 3$ = eagle's age 3 years ago

$4x - 3 = 7(x - 3)$ {three years ago, eagle was 7 times the falcon}

$4x - 3 = 7x - 21$ {used distributive property}

$4x = 7x - 18$ {added 3 to both sides}

$-3x = -18$ {subtracted $7x$ from both sides}

$x = 6$ {divided both sides by -3}

$4x = 24$ {substituted 6, in for x, into $4x$}

falcon is 6 now, eagle is 24 now

Question: 5

Farooq is 4 times as old as his niece, Selma. Ten years from now, he will be twice as old as she will be. How old is each now?

x = Selma's age now

$4x$ = Farooq's age now {Farooq is 4 times as old as Selma}

$x + 10$ = Selma's age in 10 years

$4x + 10$ = Farooq's age in 10 years

$4x + 10 = 2(x + 10)$ {his age in 10 years is twice her age in 10 years}

$4x + 10 = 2x + 20$ {used distributive property}

$2x + 10 = 20$ {subtracted $2x$ from both sides}

$2x = 10$ {subtracted 10 from both sides}

$x = 5$ {divided both sides by 2}

$4x = 20$ {substituted 5, in for x, into $4x$}

Selma is 5 now, Farooq is 20 now

Question: 6

Misbah is 9 years older than Aleem Dar. In 7 years, the sum of their ages will equal 93. Find their ages now.

x = Aleem Dar's age now

$x + 9$ = Misbah's age now {Misbah is 9 years older than Aleem Dar}

$x + 7$ = Aleem Dar's age in 7 years

$x + 16$ = Misbah's age in 7 years

$x + 7 + x + 16 = 93$ {in seven years the sum of their ages will be 93}

$2x + 23 = 93$ {combined like terms}

$2x = 70$ {subtracted 23 from both sides}

$x = 35$ {divided both sides by 35}

$x + 9 = 44$ {substituted 35, in for x, into $x + 9$}

Aleem Dar is 35, Misbah is 44

EXERCISE-II

1. A boy is 10 years older than his brother. In 4 years he will be twice as old as his brother. Find the present age of each.
2. A father is 4 times as old as his son. In 20 years the father will be twice as old as his son. Find the present age of each.
3. Pat is 20 years older than his son James. In two years Pat will be twice as old as James. How old are they now?
4. Diane is 23 years older than her daughter Amy. In 6 years Diane will be twice as old as Amy. How old are they now?
5. Fred is 4 years older than Barney. Five years ago the sum of their ages was 48. How old are they now?
6. John is four times as old as Martha. Five years ago the sum of their ages was 50. How old are they now?
7. Tim is 5 years older than JoAnn. Six years from now the sum of their ages will be 79. How old are they now?
8. Jack is twice as old as Lacy. In three years the sum of their ages will be 54. How old are they now?
9. The sum of the ages of John and Mary is 32. Four years ago, John was twice as old as Mary. Find the present age of each.
10. The sum of the ages of a father and son is 56. Four years ago the father was 3 times as old as the son. Find the present age of each.
11. The sum of the ages of a china plate and a glass plate is 16 years. Four years ago the china plate was three times the age of the glass plate. Find the present age of each plate.
12. The sum of the ages of a wood plaque and a bronze plaque is 20 years. Four years ago, the bronze plaque was one-half the age of the wood plaque. Find the present age of each plaque.
13. A is now 34 years old, and B is 4 years old. In how many years will A be twice as old as B?
14. A man's age is 36 and that of his daughter is 3 years. In how many years will the man be 4 times as old as his daughter?
15. An Oriental rug is 52 years old and a Persian rug is 16 years old. How many years ago was the Oriental rug four times as old as the Persian Rug?
16. A log cabin quilt is 24 years old and a friendship quilt is 6 years old. In how may years will the log cabin quilt be three times as old as the friendship quilt?
17. The age of the older of two boys is twice that of the younger; 5 years ago it was three times that of the younger. Find the age of each.

18. A pitcher is 30 years old, and a vase is 22 years old. How many years ago was the pitcher twice as old as the vase?
19. Marge is twice as old as Consuelo. The sum of their ages seven years ago was 13. How old are they now?
20. The sum of Jason and Mandy's age is 35. Ten years ago Jason was double Mandy's age. How old are they now?
21. A silver coin is 28 years older than a bronze coin. In 6 years, the silver coin will be twice as old as the bronze coin. Find the present age of each coin.
22. A sofa is 12 years old and a table is 36 years old. In how many years will the table be twice as old as the sofa?
23. A limestone statue is 56 years older than a marble statue. In 12 years, the limestone will be three times as old as the marble statue. Find the present age of the statues.
24. A pewter bowl is 8 years old, and a silver bowl is 22 years old. In how many years will the silver bowl be twice the age of the pewter bowl?
25. Brandon is 9 years older than Ronda. In four years the sum of their ages will be 91. How old are they now?
26. A kerosene lamp is 95 years old, and an electric lamp is 55 years old. How many years ago was the kerosene lamp twice the age of the electric lamp?
27. A father is three times as old as his son, and his daughter is 3 years younger than the son. If the sum of their ages 3 years ago was 63 years, find the present age of the father.
28. The sum of Clyde and Wendy's age is 64. In four years, Wendy will be three times as old as Clyde. How old are they now?
29. The sum of the ages of two ships is 12 years. Two years ago, the age of the older ship was three times the age of the newer ship. Find the present age of each ship.
30. Chelsea's age is double Daniel's age. Eight years ago the sum of their ages was 32. How old are they now?

ANSWERS

1.	6, 16	2.	10, 40	3.	18, 38	4.	17, 40	5.	27, 31	6.	12, 48	7.	31, 36
8.	16, 32	9.	12, 20	10.	40, 16	11.	10, 6	12.	12, 8	13.	26	14.	8
15.	4	16.	3	17.	10, 20	18.	14	19.	9, 18	20.	15, 20	21.	50, 22
22.	12	23.	72, 16	24.	6	25.	37, 46	26.	15	27.	45	28.	14, 54
29.	8, 4	30.	16, 32										

DISTANCE, TIME AND RATE RELATED PROBLEMS

Distance word problems, often also called "uniform rate" problems, involve something travelling at some fixed and steady ("uniform") pace ("rate" or "speed"), or else moving at some average speed. Whenever you read a problem that involves "how fast", "how far", or "for how long", you should think of the distance equation d $= rt$, where d stands for distance, r stands for the (constant or average) rate of speed, and t stands for time.

The basic formula used in all rate problems is

Distance = rate × time or d = rt

Of course, you can solve for one of the other variables:

$$\text{rate} = \frac{\text{distance}}{\text{time}} \quad \text{or} \quad r = \frac{d}{t}$$

$$\text{time} = \frac{\text{distance}}{\text{rate}} \quad \text{or} \quad t = \frac{d}{r}$$

Warning: Make sure that the units for time and distance agree with the units for the rate. For instance, if they give you a rate of feet per second, then your time must be in seconds and your distance must be in feet. Sometimes they try to trick you by using the wrong units, and you have to catch this and convert to the correct units.

Example 1. A 555-mile, 5-hour plane trip was flown at two speeds. For the first part of the trip, the average speed was 105 mph. Then the tailwind picked up, and the remainder of the trip was flown at an average speed of 115 mph. For how long did the plane fly at each speed?

First of all set up a grid:

	d	r	t
First part	d	105	t
Second part	555 – d	115	5 – t
Total	555	---	5

Using "d = rt", the first row gives me d = 105t and the second row gives me:

$$555 - d = 115(5 - t)$$

Since the two distances add up to 555, add the two distance expressions, and set their sum equal to the given total:

$$555 = 105t + 115(5 - t)$$

Then solve:

$$555 = 105t + 575 - 115t$$
$$555 = 575 - 10t$$
$$-20 = -10t$$
$$2 = t$$

According to my grid, "t" stands for the time spent on the first part of the trip, so my answer is "The plane flew for two hours at 105 mph and three hours at115 mph."

Example 2. Calculate the speed of a train which covers a distance of 150 km in 3 hours.

Solution:

$$\text{Speed} = \frac{\text{Distance covered}}{\text{Time taken}} = \frac{150}{3} = 50 \text{ km/hr.}$$

Example 3. How long does a train 100 meters long running at the rate of 40 km/hr take to cross a telegraphic pole?

Solution:

In crossing the pole, the train must travel its own length.

∴ Distance travelled is 100 meters.

$$\text{Speed} = 40 \text{ km/hr} = \frac{40\times1000}{60\times60} = \frac{100}{9}\text{ m/sec}.$$

$$\therefore \text{Time taken to cross the pole} = \frac{100}{100/9} = 9 \text{ seconds}$$

Example 4. A train running at a speed of 90 km/hr passes a pole on the platform in 20 seconds. Find the length of the train in meters.

Solution:

Speed of the train = 90 km/hr

$$= 90\times\frac{5}{18} = 25 \text{ m/sec}.$$

∴ Length of the train = Speed of the train × time taken in crossing the pole

$= 25 \times 20 = 500$ m.

EXERCISE

1. A is 60 miles from B. An automobile at A starts for B at the rate of 20 miles an hour at the same time that an automobile at B starts for A at the rate of 25 miles an hour. How long will it be before the automobiles meet?
2. Two automobiles are 276 miles apart and start at the same time to travel toward each other. They travel at rates differing by 5 miles per hour. If they meet after 6 hours, find the rate of each.
3. Two trains travel toward each other from points which are 195 miles apart. They travel at rate of 25 and 40 miles an hour respectively. If they start at the same time, how soon will they meet?
4. A and B start toward each other at the same time from points 150 miles apart. If A went at the rate of 20 miles an hour, at what rate must B travel if they meet in 5 hours?
5. A passenger and a freight train start toward each other at the same time from two points 300 miles apart. If the rate of the passenger train exceeds the rate of the freight train by 15 miles per hour, and they meet after 4 hours, what must the rate of each be?
6. Two automobiles started at the same time from a point, but traveled in opposite directions. Their rates were 25 and 35 miles per hour respectively. After how many hours were they 180 miles apart?
7. A man having ten hours at his disposal made an excursion, riding out at the rate of 10 miles an hour and returning on foot, at the rate of 3 miles an hour. Find the distance he rode.

8. A man walks at the rate of 4 miles per hour. How far can he walk into the country and ride back on a trolley that travels at the rate of 20 miles per hour, if he must be back home 3 hours from the time he started?

9. A boy rides away from home in an automobile at the rate of 28 miles an hour and walks back at the rate of 4 miles an hour. The round trip requires 2 hours. How far does he ride?

10. A motorboat leaves a harbor and travels at an average speed of 15 mph toward an island. The average speed on the return trip was 10 mph. How far was the island from the harbor if the total trip took 5 hours?

11. A family drove to a resort at an average speed of 30 mph and later returned over the same road at an average speed of 50 mph. Find the distance to the resort if the total driving time was 8 hours.

12. As part of his flight trainging, a student pilot was required to fly to an airport and then return. The average speed to the airport was 90 mph, and the average speed returning was 120 mph. Find the distance between the two airports if the total flying time was 7 hours.

13. A, who travels 4 miles an hour starts from a certain place 2 hours in advance of B, who travels 5 miles an hour in the same direction. How many hours must B travel to overtake A?

14. A man travels 5 miles an hour. After traveling for 6 hours another man starts at the same place, following at the rate of 8 miles an hour. When will the second man overtake the first?

15. A motorboat leaves a harbor and travels at an average speed of 8 mph toward a small island. Two hours later a cabin cruiser leaves the same harbor and travels at an average speed of 16 mph toward the same island. In how many hours after the cabin cruiser leaves will the cabin cuiser be alongside the motorboat?

16. A long distance runner started on a course running at an average speed of 6 mph. One hour later, a second runner began the same course at an average speed of 8 mph. How long after the second runner started will the second runner overtake the first runner?

17. A car traveling at 48 mph overtakes a cyclist who, riding at 12 mph, has had a 3 hour head start. How far from the starting point does the car overtake the cyclist?

18. A jet plane traveling at 600 mph overtakes a propeller-driven plane which has had a 2 hour head start. The propeller-driven plane is traveling at 200 mph. How far from the starting point does the jet overtake the propeller-driven plane?

19. Two men are traveling in opposite directions at the rate of 20 and 30 miles an hour at the same time and from the same place. In how many hours will they be 300 miles apart?

20. Running at an average rate of 8 m/s, a sprinter ran to the end of a track and then jogged back to the starting point at an average rate of 3 m/s. The sprinter took 55 s to run to the end of the track and jog back. Find the length of the track.

21. A motorboat leaves a harbor and travels at an average speed of 18 mph to an island. The average speed on the return trip was 12 mph. How far was the island from the harbor if the total trip took 5 h?

22. A motorboat leaves a harbor and travels at an average speed of 9 mph toward a small island. Two hours later a cabin cruiser leaves the same harbor and travels at an average speed of 18 mph toward the same island. In how many hours after the cabin cruiser leaves will the cabin cruiser be alongside the motorboat?

23. A jet plane traveling at 570 mph overtakes a propeller-driven plane that has had a 2 h head start. The propeller-driven plane is traveling at 190 mph. How far from the starting point does the jet overtake the propeller-driven plane?

24. Two trains start at the same time from the same place and travel in opposite directions. If the rate of one is 6 miles per hour more than the rate of the other and they are 168 miles apart at the end of 4 hours, what is the rate of each?

25. As part of flight traning, a student pilot was required to fly to an airport and then return. The average speed on the way to the airport was 100 mph, and the average speed returning was 150 mph. Find the distance between the two airports if the total flight time was 5 h.

ANSWERS

1.	$1\frac{1}{3}$	**2.**	25.5, 20.5	**3.**	3	**4.**	10	**5.**	30, 45	**6.**	3	**7.**	300/13
8.	10	**9.**	7	**10.**	30	**11.**	150	**12.**	360	**13.**	8	**14.**	10
15.	2	**16.**	3	**17.**	48	**18.**	600	**19.**	6	**20.**	120	**21.**	36
22.	2	**23.**	570	**24.**	24, 18	**25.**	300						

MULTIPLE CHOICE QUESTIONS WITH EXPLANATORY ANSWERS

In each of the following questions a number of possible answers are given, out of which one answer is correct. Find out the correct answer.

1. How many seconds longer does it take to drive 1 mile at 40 miles per hour than at 60 miles per hour?

(a) 15 **(b)** 30 **(c)** 40 **(d)** 60

Ans. (b)

Solution: Since $d = rt$, $t = \frac{d}{r}$. The time required to drive 1 mile at 40 miles per hour is

$t = \frac{1 \text{ mile}}{40 \text{ miles per hour}} = \frac{1}{40}\text{hour} = \left(\frac{1}{40}\right)(60)$ minutes = 1.5 minutes = (1.5) (60) seconds = 90 seconds.

The time required to drive 1 mile at 60 miles per hour is $t = \frac{1 \text{ mile}}{60 \text{ miles per hour}} = \frac{1}{60}\text{hour} = \left(\frac{1}{60}\right)(60)$ minutes = 1 minute = 60 seconds. So it takes 90 – 60 = 30 seconds more to drive 1 mile at 40 miles per hour than at 60 miles per hour.

2. **Ali was 40 years old when his son Asim was born. How old was Ali when he was 5 times as old as Asim?**

(a) 40 (b) 45 (c) 50 (d) 60

Ans. (c)

Solution: x years after Asim was born, Ali was $40 + x$ years old and Asim was x years old.

$$40 + x = 5x$$
$$\Rightarrow \quad 40 = 4x$$
$$\Rightarrow \quad x = 10$$

So Ali was 50 years old (and Asim was 10).

3. **The length of each side of a square is 3 more than the length of each side of a regular pentagon. If the perimeters of the square and pentagon are equal, how long is each side of the pentagon?**

(a) 6 (b) 12 (c) 10 (d) 15

Ans. (b)

Solution: Let x = the length of each side of the pentagon. Then $x + 3$ = the length of each side of the square. The perimeter of the pentagon is $5x$ and the perimeter of the square is $4(x + 3)$. Therefore:

$$5x = 4(x + 3)$$
$$5x = 4x + 12$$
$$5x - 4x = 12$$
$$\Rightarrow \quad x = 12$$

4. **A jar contains only red, white, and blue marbles. The number of red marbles is 4/5 the number of white ones, and the number of white ones is 3/4 the number of blue ones. If there are 470 marbles in all, how many of them are blue?**

(a) 120 (b) 135 (c) 150 (d) 200

Ans. (d)

Solution: If b is the number of blue marbles, then there are $\frac{3}{4}$b white ones, and $\frac{4}{5}\left(\frac{3}{4}b\right) = \frac{3}{5}b$ red ones. Therefore,

$$470 = b + \frac{3}{4}b + \frac{3}{5}b = b\left(1 + \frac{3}{4} + \frac{3}{5}\right) = \frac{47}{20}b$$

$$\text{So,} \quad b = 470 \div \frac{47}{20} = \overset{10}{\cancel{470}} \times \frac{20}{\underset{1}{\cancel{47}}} = 200$$

5. **As a fund-raiser, the school band was selling two types of candy: lollipops for 40 cents each and chocolate bars for 75 cents each. One Monday, they sold 150 candies and raised 74 dollars. How many lollipops did they sell?**

(a) 75 (b) 90 (c) 96 (d) 110

Ans. (d)

Solution: Let x represent the number of chocolate bars sold; then $150 - x$ is the number of lollipops sold. We must use the same units, so we could write 75 cents as .75 dollars or 74 dollars as 7400 cents. Let's avoid the decimals: x chocolates sold for $75x$ cents and $(150 - x)$ lollipops sold for $40(150 - x)$ cents. So

$$7400 = 75x + 40\,(150 - x)$$

$$75x + 6000 - 40x = 6000 + 35x$$

$$\Rightarrow \quad 1400 = 35x$$

$$\Rightarrow \quad x = 40 \text{ and } 150 - 40 = 110$$

6. **Compare the speed of two trains, one moving at the speed of 80 km/hr and the other at 10m/sec.**

(a) 30 : 9 (b) 40 : 9 (c) 20 : 9 (d) None of these

Ans. (c)

Solution: 80 km/hr means $\left(80 \times \frac{5}{18}\right)$ m/sec.

$\therefore$ Required comparison is $80 \times \frac{5}{18}$: 10 or, 20 : 9

7. **A train 100 metres long passes a bridge at the rate of 72 km/hr in 25 seconds. What is the length of the bridge?**

(a) 400 m (b) 170 m (c) 600 m (d) None of these

Ans. (a)

Solution: Distance travelled by the train in 25 seconds at 72 km/hr

$$= 72 \times \frac{5}{18} \times 25 = 500\text{m}$$

$\therefore$ Length of the Bridge = 500 – length of train

= 500 – 100 = 400 m

8. **A train travels 225 km in 3.5 hours and 370 km in 5 hours. Find the average speed of train.**

(a) 80 km/hr (b) 60 km/hr (c) 70 km/hr (d) None of these

Ans. (c)

Solution: Here, $x_1 = 225$, $x_2 = 370$, $T_1 = 3.5$ and $T_2 = 5$.

$$\therefore \quad \text{Average speed of train} = \frac{x_1 + x_2}{T_1 + T_2}$$

$$\therefore \quad \text{Average speed of train} = = \frac{225 + 370}{3.5 + 5}$$

$$\therefore \quad \text{Average speed of train} = = 70 \text{ km/hr}$$

9. **A father is twice as old as his son. 20 years ago, the age of the father was 12 times the age of the son. The present age of the father (in years) is:**

(a) 44 (b) 32 (c) 22 (d) 45 (e) 50

Ans. **(a)**

Solution: Let son's age = x. Then, father's age = $2x$.

$$12(x - 20) = (2x - 20) \Leftrightarrow x = 22$$

$\therefore$ Father's present age = 44 years

10. **Ten years ago A was half of B in age. If the ratio of their present ages is 3 : 4, what will be the total of their present ages?**

(a) 8 years **(b)** 20 years **(c)** 30 years **(d)** 35 years

Ans. **(d)**

Solution: Let A's age 10 years ago = x. Then, B's age 10 years ago = $2x$

$$\frac{x+10}{2x+10} = \frac{3}{4} \Leftrightarrow 4(x + 10) = 3(2x + 10) \Leftrightarrow x = 5$$

Total of their present ages = $(x + 10 + 2x + 10)$

$= (3x + 20) = 35$ years

CHAPTER 30

SAMPLING

Sampling is concerned with the selection of a subset of individuals from within a statistical population to estimate characteristics of the whole population. Each observation measures one or more properties (such as weight, location, and colour) of observable bodies distinguished as independent objects or individuals.

Since it is generally impossible to study an entire population (every individual in a country, all college students, every geographic area, etc.), researchers typically rely on sampling to acquire a section of the population to perform an experiment or observational study. It is important that the group selected be representative of the population, and not biased in a systematic manner. For example, a group comprised of the wealthiest individuals in a given area probably would not accurately reflect the opinions of the entire population in that area. For this reason, randomization is typically employed to achieve an unbiased sample.

Probability and Non Probability Sampling

A **probability sampling** scheme is one in which every unit in the population has a chance (greater than zero) of being selected in the sample, and this probability can be accurately determined. Methods include random sampling, systematic sampling, and stratified sampling.

Non probability sampling is a sampling method where some elements of population have *no* chance of selection or where the probability of selection can't be accurately determined. It involves the selection of elements based on assumptions regarding the population of interest, which forms the criteria for selection. Methods include convenience sampling, judgment sampling, quota sampling, and snowball sampling.

PROBABILITY SAMPLING

Simple Random Sampling

Simple random sampling is the basic sampling technique where we select a group of subjects (a sample) for study from a larger group (a population). Each individual is chosen entirely by chance and each member of the population has an equal chance of being included in the sample. Every possible sample of a given size has the same chance of selection.

Stratified Random Sampling

A stratified sample is obtained by taking samples from each stratum or sub-group of a population. When we sample a population with several strata, we generally require that the proportion of each stratum in the sample should be the same as in the population. Stratified sampling techniques are generally used when the population is heterogeneous, or dissimilar, where certain homogeneous, or similar, sub-populations can be isolated (strata). Simple random sampling is most appropriate when the entire population from which the sample is taken is homogeneous. Some reasons for using stratified sampling over simple random sampling are:

(a) the cost per observation in the survey may be reduced;

(b) estimates of the population parameters may be wanted for each sub-population;

(c) increased accuracy at given cost.

Suppose a farmer wishes to work out the average milk yield of each cow type in his herd which consists of Sahiwal, Friesian, Red Sindhi and Dhanni. He could divide up his herd into the four sub-groups and take samples from these. It is stratified random sampling.

Cluster Sampling

Cluster sampling is very different from stratified sampling. With cluster sampling one should divide the population into groups (clusters); obtain a simple random sample from all possible clusters; and obtain data on every sampling unit in each of the randomly selected clusters. It is important to note that, unlike with the strata in stratified sampling, the clusters should be microcosms, rather than subsections, of the population. Each cluster should be heterogeneous. Additionally, the statistical analysis used with cluster sampling is not only different, but also more complicated than that used with stratified sampling.

Examples of clusters:

- City blocks – political or geographical
- Hospitals – illnesses
- Automobile – set of four tires

Multi-stage Random Sampling

A *multi-stage random sample* is constructed by taking a series of simple random samples in stages. This type of sampling is often more practical than simple random sampling for studies requiring "on location" analysis, such as door-to-door surveys. In a multistage random sample, a large area, such as a country, is first divided into smaller regions (such as provinces), and a random sample of these regions is collected. In the second stage, a random sample of smaller areas (such as districts) is taken from within each of the regions chosen in the first stage. Then, in the third stage, a random sample of even smaller areas (such as towns/tehsils) is taken from within each of the areas chosen in the second stage. If these areas are sufficiently small for the purposes of the study, then the researcher might stop at the third stage. If not, he or she may continue to sample from the areas chosen in the third stage, etc., until appropriately small areas have been chosen.

NON PROBABILITY SAMPLING

Convenient Sampling

Sometimes known as **grab** or **opportunity sampling or accidental or haphazard sampling**. A type of nonprobability sampling which involves the sample being drawn from that part of the population which is close to hand. That is, readily available and convenient. The researcher using such a sample cannot scientifically make generalizations about the total population from this sample because it would not be representative enough. This type of sampling is most useful for pilot testing.

Judgmental sampling or Purposive sampling

The researcher chooses the sample based on who they think would be appropriate for the study. This is used primarily when there are a limited number of people that have expertise in the area being researched.

Snowball Sampling

Snowball sampling is a special non-probability method used when the desired sample characteristic is rare. It may be extremely difficult or cost prohibitive to locate respondents in these situations. This technique relies on referrals from initial subjects to generate additional subjects. It lowers search costs; however, it introduces bias because the technique itself reduces the likelihood that the sample will represent a good cross section from the population.

ADVANTAGES AND DISADVANTAGES OF SAMPLING

Advantages of Sampling

Sampling ensures convenience, collection of intensive and exhaustive data, suitability in limited resources and better rapport. In addition to this, sampling has the following advantages.

1. **Cost of sampling:** If data were to be collected for the entire population, the cost will be quite high. A sample is a small proportion of a population. So, the cost will be lower if data is collected for a sample of population which is a big advantage.

2. **Time consumption:** Use of sampling takes less time also. It consumes less time than census technique. Tabulation, analysis, etc. take much less time in the case of a sample than in the case of a population.

3. **Scope of sampling:** The investigator is concerned with the generalization of data. To study a whole population in order to arrive at generalizations would be impractical. The process of sampling makes it possible to arrive at generalizations by studying the variables within a relatively small proportion of the population.

4. **Accuracy of data:** A sample represents the population from which its is drawn. It permits a high degree of accuracy due to a limited area of operations. Moreover, careful execution of field work is possible. Ultimately, the results of sampling studies turn out to be sufficiently accurate.

5. **Organization of convenience:** Organizational problems involved in sampling are very few. Since sample is of a small size, vast facilities are not required. Sampling is therefore economical in respect of resources. Study of samples involves less space and equipment.

6. **Intensive and exhaustive data:** In sample studies, measurements or observations are made of a limited number. So, intensive and exhaustive data is collected.

7. **Suitable in limited resources:** The resources available within an organization may be limited. Studying the entire universe is not viable. The population can be satisfactorily covered through sampling. Where limited resources exist, use of sampling is an appropriate strategy while conducting marketing research.

Disadvantages of Sampling

The reliability of the sample depends upon the appropriateness of the sampling method used. The purpose of sampling theory is to make sampling more efficient. But the real difficulties lie in selection, estimation and administration of samples.

1. **Chances of bias:** The serious limitation of the sampling method is that it involves biased selection and thereby leads us to draw erroneous conclusions. Bias arises when the method of selection of sample employed is faulty. Relative small samples properly selected may be much more reliable than large samples poorly selected.

2. **Difficulties in selecting a truly representative sample:** Selection of a truly representative sample is difficult when the phenomena under study are of a complex nature. Selecting good samples is difficult.

3. **Inadequate knowledge in the subject:** Use of sampling method requires adequate subject specific knowledge in sampling technique. Sampling involves statistical analysis and calculation of probable error. When the researcher lacks specialized knowledge in sampling, he may commit serious mistakes. Consequently, the results of the study will be misleading.

4. **Changeability of units:** When the units of the population are not in homogeneous, the sampling technique will be unscientific. In sampling, though the number of cases is small, it is not always easy to stick to the selected cases. The units of sample may be widely dispersed.

Some of the cases of sample may not cooperate with the researcher and some others may be inaccessible. Because of these problems, all the cases may not be taken up. The selected cases may have to be replaced by other cases. Changeability of units stands in the way of results of the study and it affects the outcome.

5. **Impossibility of sampling:** Deriving a representative sample is difficult, when the universe is too small or too heterogeneous. In this case, census study is the only alternative. Moreover, in studies requiring a very high standard of accuracy, the sampling method may be unsuitable. There will be chances of errors even if samples are drawn most carefully.

MULTIPLE CHOICE QUESTIONS (MCQS)

1. **Random sampling error.**

   (a) Does not occur in nonprobability samples.

   (b) Results from the nature of a study's design and the inappropriate or random administration of the sampling process.

   (c) Is a technical term that applies only to simple random sampling.

   (d) Is a function of sample size.

2. **Name a sampling procedure in which initial respondents are selected by non probability methods, and then additional respondents are obtained from information provided by initial respondents.**

   (a) Semi-random **(b)** Initial **(c)** Quota **(d)** Snowball

3. **A researcher divides the population of product users into three groups based on degree of use. If the researcher then draws a random sample from each user group independently, she has created a _____ sample.**

   (a) Random **(b)** Stratified **(c)** Judgment **(d)** Group data

4. **Stratified samples may be distinguished from quota samples because with a stratified sample, sample elements are selected.**

   (a) Randomly **(b)** Judgmentally **(c)** Proportionately **(d)** Sequentially

5. **If a researcher wishing to draw a sample from sequentially numbered invoices uses a random starting point, then draws every 50th invoice, he has thus drawn a _____ sample.**

(a) Simple random (b) Sequential (c) Stratified (d) Systematic

6. **Suppose a researcher is concerned with a nominal scale that identifies users versus nonusers of bank credit cards. The measure of central tendency appropriate to this scale is the.**

(a) Mean (b) Median (c) Mode (d) Average

7. **A specific range of numbers within which a population mean should lie is.**

(a) The range (b) The confidence coefficient

(c) The confidence interval (d) The confidence level

8. **Which of the following factors is *not* necessary in determining sample size?**

(a) Estimated standard deviation of the population

(b) Magnitude of acceptable error

(c) Confidence level

(d) All of these are necessary

9. **When each member of a population has an equally likely chance of being selected, this is called.**

(a) A nonrandom sampling method (b) A quota sample

(c) A snowball sample

(d) Probability selection method

10. **Interviewing all members of a given population is called.**

(a) A sample (b) A Gallup poll

(c) A census (d) A Nielsen audit

ANSWERS

1.	d	**2.**	d	**3.**	b	**4.**	a	**5.**	d	**6.**	c	**7.**	c
8.	d	**9.**	d	**10.**	c								

CHAPTER 31

MENTAL ABILITY SCALES

General mental ability (GMA) is a term used to describe the level at which an individual learns, understands instructions, and solves problems. Tests of general mental ability include scales that measure specific abilities such as *verbal, mechanical, numerical, social,* and *spatial ability.* The overall score is considered the most important factor, explaining more variation in individual performance than specific abilities.

An **aptitude** is the ability to learn or to develop proficiency in an area (if provided with appropriate education or training). It is like talent. Examples are various types of reasoning, artistic ability, motor coordination, musical talent. There are aptitude tests that measure mechanical and linguistic ability, as well as more specific skills, such as military flight and computer programming.

Intelligence is a general mental capability that involves the ability to reason, plan, solve problems, think abstractly, comprehend ideas and language, and learn. Intellectual ability involves comprehension; understanding, and learning from experience. Intelligence tests are aimed at assessing a person's underlying intellectual ability.

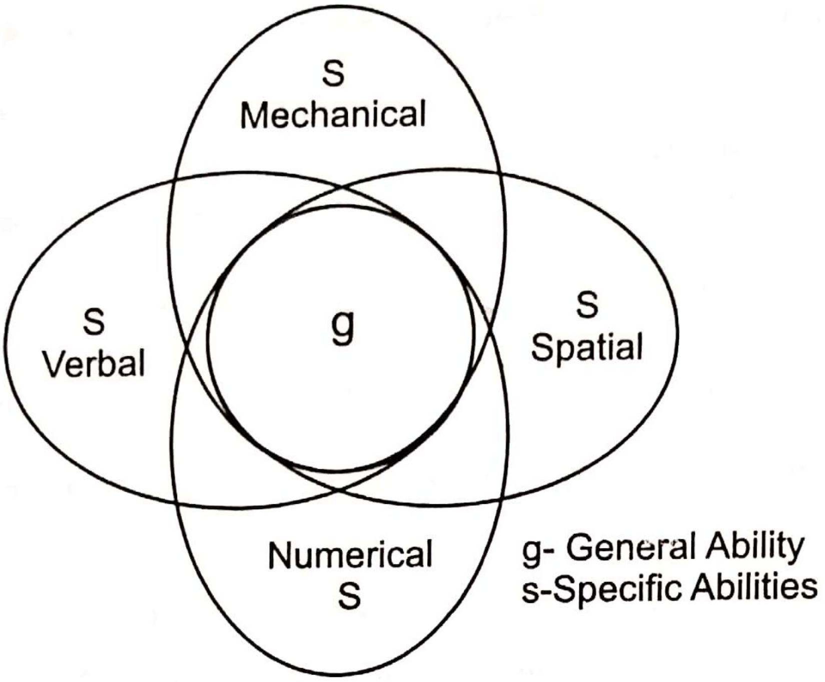
A typical dictionary definition of intelligence is the capacity to acquire and apply knowledge. Intelligence includes the ability to benefit from experience, act purposefully, solve problems, and adapt to new situations. There is a long history of disagreement about what actually constitutes intelligence.

EARLY THEORIES OF INTELLIGENCE

Charles Spearman believed intelligence is general. People who are bright in one area are usually bright in other areas as well. Charles Spearman proposed a **general intelligence factor**, g which underlies all intelligent behaviour.

According to Spearman, intelligence consist only two factor General factor (g) and Specific factor (s). **'g' factor** is general mental ability towards different tasks. **'s' factor** is specific capacity that helps the person to deal with specific problems. 'g' factor remains constant while 's' factor varies with the intellectual activity. For example, One's performance in physics = g+s1; Maths = g+s2; English = g+s3. 's' factor influenced by education and training whereas 'g' is innate.

In 1938, **Thurstone** challenged the idea of a single intelligence, such as the idea supported by Spearman and Galton and suggested there were different types of intelligence, and there was no one type of intelligence that was ahead of others. He came up with the idea of seven distinct **primary mental abilities:** *spatial ability, perceptual speed, numerical ability, verbal meaning, memory, word fluency,* and *reasoning.*



PRIMARY AND SECONDARY MENTAL ABILITIES

Primary mental abilities are groups of related intellectual skills, whereas secondary mental abilities are broader intellectual skills that include and organize the primary abilities. Secondary mental abilities include fluid and crystallized intelligence. **R. B. Cattell** identified clusters of mental abilities i.e. Crystallized intelligence and Fluid intelligence

Crystallized intelligence includes abilities such as reasoning and verbal skills. Crystallized intelligence is the knowledge that one has acquired through life experience and education. This knowledge includes the comprehension of communication, judgment, and the ability to remember many other things.

Fluid intelligence includes skills such as spatial and visual imagery, rote memory, and the ability to notice visual details. Fluid intelligence stresses the abilities that make an individual a flexible thinker. Sudoku, cross words, word searches and mazes are way to test fluid intelligence. While education can increase crystallized intelligence, it was not thought to have any effect on fluid intelligence.

EIGHT TYPES OF INTELLIGENCE

Psychologist **Howard Gardner** proposed the idea of eight kind of intelligence, which are relatively independent of one another. These eight types of intelligence are:

1. **Linguistic:** spoken and written language skills
2. **Logical–mathematical:** number skills
3. **Musical:** performance or composition skills
4. **Spatial:** ability to evaluate and analyze the visual world
5. **Bodily-kinesthetic:** dance or athletic abilities
6. **Interpersonal:** skill in understanding and relating to others
7. **Intrapersonal:** skill in understanding the self
8. **Nature:** skill in understanding the natural world

Gardner believes that each of these domains of intelligence has inherent value but that culture and context may cause some domains to be emphasized over others. Critics of the idea of multiple intelligences maintain that these abilities are talents rather than kinds of intelligence.

Triarchic Theory of Intelligence

In the 1980s and 1990s, **Robert Sternberg** proposed a **Triarchic Theory of Intelligence** that distinguishes among three aspects of intelligence:

- **Analytical intelligence** includes the ability to learn how to do things, solve problems, and acquire new knowledge.
- **Creative intelligence** includes the ability adjust to new tasks, use new concepts, and respond well in new situations.
- **Practical intelligence** includes the ability to select contexts in which you can excel and solve practical problems.

EMOTIONAL INTELLIGENCE

Some researchers distinguish **emotional intelligence** as an ability that helps people to perceive, express, understand, and regulate emotions. Other researchers maintain that this ability is a collection of personality traits such as empathy and extroversion, rather than a kind of intelligence. **Daniel Goleman** proposed theory of Emotional Intelligence. Emotional intelligence has five components

- Knowing one's own emotions
- Managing one's own emotions
- Using emotions to motivate oneself
- Recognizing the emotions of other people
- Managing relationships

INTELLIGENCE TESTS

The **psychometric approach** to intelligence emphasizes people's performance on standardized aptitude tests. **Aptitude tests** predict people's future ability to acquire skills or knowledge. **Achievement tests,** on the other hand, measure skills and knowledge that people have already learned.

Types of Intelligence Tests

Intelligence tests can be given individually or to groups of people. The best-known individual intelligence tests are the Binet - Simon scale, the Stanford-Binet Intelligence Scale, and the Wechsler Adult Intelligence Scale.

The Binet-Simon Scale

Alfred Binet and his colleague Theodore Simon devised this general test of mental ability in 1905, and it was revised in 1908 and 1911. The test yielded scores in terms of mental age. **Mental age** is the chronological age that typically corresponds to a particular level of performance.

Example: A ten-year-old child whose score indicates a mental age of twelve performed like a typical twelve-year-old.

The Stanford-Binet Intelligence Scale

In 1916, **Lewis Terman** and his colleagues at Stanford University created the Stanford-Binet Intelligence Scale by expanding and revising the Binet-Simon scale. The Stanford-Binet yielded scores in terms of intelligence quotients.

The **intelligence quotient (IQ)** is the mental age divided by the chronological age and multiplied by 100. IQ scores allowed children of different ages to be compared.

Example: A ten-year-old whose performance resembles that of a typical twelve-year-old has an IQ of 120 (12 divided by 10 times 100).

There are two problems with the intelligence quotient approach:

1. The score necessary to be in the top range of a particular age group varies, depending on age.
2. The scoring system had no meaning for adults. For example, a fifty-year-old man who scores like a thirty-year-old can't accurately be said to have low intelligence.

Wechsler Adult Intelligence Scale

David Wechsler published the first test for assessing intelligence in adults in 1939. The Wechsler Adult Intelligence Scale contains many items that assess nonverbal reasoning ability and therefore depends less on verbal ability that does the Stanford-Binet. It also provides separate scores of verbal intelligence and nonverbal or performance intelligence, as well as a score that indicates overall intelligence.

The term intelligence quotient, or *IQ*, is also used to describe the score on the Wechsler test. However, the Wechsler test presented scores based on a normal distribution of data rather than the intelligence quotient. The **normal distribution** is a symmetrical bell-shaped curve that represents how characteristics like IQ are distributed in a large population. In this scoring system, the mean IQ score is set at 100, and the standard deviation is set at 15. The test is constructed so that about two-thirds of people tested (68 %) will score within one standard deviation of the mean, or between 85 and 115.

On the Wechsler test, the IQ score reflects where a person falls in the normal distribution of IQ scores. Therefore, this score, like the original Stanford-Binet IQ score, is a relative score, indicating how the test taker's score compares to the scores of other people. Most current intelligence tests, including the revised versions of the Stanford-Binet, now have scoring systems based on the normal distribution. About 95 % of the population will score between 70 and 130 (within two standard deviations from the mean), and about 99.7 % of the population will score between 55 and 145 (within three standard deviations from the mean).

CHARACTERISTICS OF IQ TESTS

Some characteristics of IQ tests are standardization, norms, percentile scores, standardization samples, reliability, and validity.

Standardization

Intelligence tests are **standardized**, which means that uniform procedures are used when administering and scoring the tests. Standardization helps to ensure that people taking a particular test all do so under the same conditions. Standardization also allows test takers to be compared, since it increases the likelihood that any difference in scores between test takers is due to ability rather than the testing environment.

Norms and Percentile Scores

Researchers use norms when scoring the tests. **Norms** provide information about how a person's test score compares with the scores of other test takers. Norms allow raw test scores to be converted into percentile scores.

A **percentile score** indicates the percentage of people who achieved the same as or less than a particular score. For example, if someone answered twenty items correctly on a thirty-item vocabulary test, he receives a raw score of 20. He consults the test norms and finds that a raw score of 20 corresponds with a percentile score of 90. This means that he scored the same as or higher than 90 % of people who took the same test.

Standardization Samples

Psychologists come up with norms by giving a test to a standardization sample. A **standardization sample** is a large group of people that is representative of the entire population of potential test takers.

Reliability

Most intelligence tests have good reliability. **Reliability** is a test's ability to yield the same results when the test is administered at different times to the same group of people.

Validity

Validity is a test's ability to measure what it is supposed to measure. Although intelligence tests cannot be considered good measures of general intelligence or general mental ability, they are reasonably valid indicators of the type of intelligence that enables good academic performance.

FACTORS AFFECTING INTELLIGENCE

Today, researchers generally agree that heredity and environment have an interactive influence on intelligence. Many researchers believe that there is a **reaction range** to IQ, which refers to the limits placed on IQ by heredity. Heredity places an upper and lower limit on the IQ that can be attained by a given person. The environment determines where within these limits the person's IQ will lie. Despite the prevailing view that both heredity and environment influence intelligence, researchers still have different opinions about how much each contributes and how they interact.

Hereditary Influences: Evidence for hereditary influences on intelligence comes from the following observations:

- Family studies show that intelligence tends to run in families.

- Twin studies show a higher correlation between identical twins in IQ than between fraternal twins. This holds true even when identical twins reared apart are compared to fraternal twins reared together.
- Adoption studies show that adopted children somewhat resemble their biological parents in intelligence.

Heritability is a mathematical estimate that indicates how much of a trait's variation in a population can be attributed to genes. Estimates of the heritability of intelligence vary, depending on the methods used. Most researchers believe that heritability of intelligence is between 60 % and 80 %.

Environmental Influences: Evidence for environmental influences on intelligence comes from the following observations:

- Adoption studies demonstrate that adopted children show some similarity in IQ to their adoptive parents.
- Adoption studies also show that siblings reared together are more similar in IQ than siblings reared apart. This is true even when identical twins reared together are compared to identical twins reared apart.
- Biologically unrelated children raised together in the same home have some similarity in IQ.
- IQ declines over time in children raised in deprived environments, such as understaffed orphanages or circumstances of poverty and isolation. Conversely, IQ improves in children who leave deprived environments and enter enriched environments.
- People's performance on IQ tests has improved over time in industrialized countries. This strange phenomenon, which is known as the **Flynn effect**, is attributed to environmental influences. It cannot be due to heredity, because the world's gene pool could not have changed in the seventy years or so since IQ testing began.

The precise cause for the Flynn effect is unclear. Researchers speculate that it may be due to environmental factors such as decreased prevalence of severe malnutrition among children, enhancing of skills through television and video games, improved schools, smaller family sizes, higher level of parental education, or improvements in parenting.

Race and Nationality: Studies have shown a discrepancy in average IQ scores between whites and minority groups in the United States. Black, Native American, and Hispanic people score lower, on average, than white people on standardized IQ tests. Controversy exists about whether this difference is due to heredity or environment.

Sex and intelligence: Most studies show that despite sometimes significant differences in subtest scores, men and women have the same average IQ. Women perform better on tests of memory and verbal proficiency for example, while men perform better on tests of mathematical and spatial ability. Although gender-related differences in average IQ are insignificant, male scores display a higher variance: there are more men than women with both very high and very low IQs.

Age: A person who bright or dull in childhood tends to remain bright or dull throughout his life. Growth I intelligence can continue through the early twenties, person achieves his maximum 10 at about 20 years are and remain stable till 10 years and decreases after that. Some abilities remain constant while others decline rapidly due to decline in physical efficiency. It is observed that the more intelligent person has more rapid of mental growth and continues to develop longer than does the mentally less able.

Health and Physical Development: Physical and mental health is related to one's ability to gain desired achievement. A mental activity with delicate health, one may not possess enough energy to engage in mental activity to the extent that he achieves success. Physical defects such as in complete maturation of brain cells sensory and physical handicaps many interfere with observable intelligent behavior. Diseases also affect intelligence. Emotional bodies may interferes one's ability. Unfavorable health affects mental status of the individual.

Social and Economic Conditions: Home plays a significance role in the early developmental years. Home conditions influence on behavior and attitude. The financial status of the parents, neighborhood and environmental conditions also effect on the intelligence but due to their socio-economic status.

* * * * *